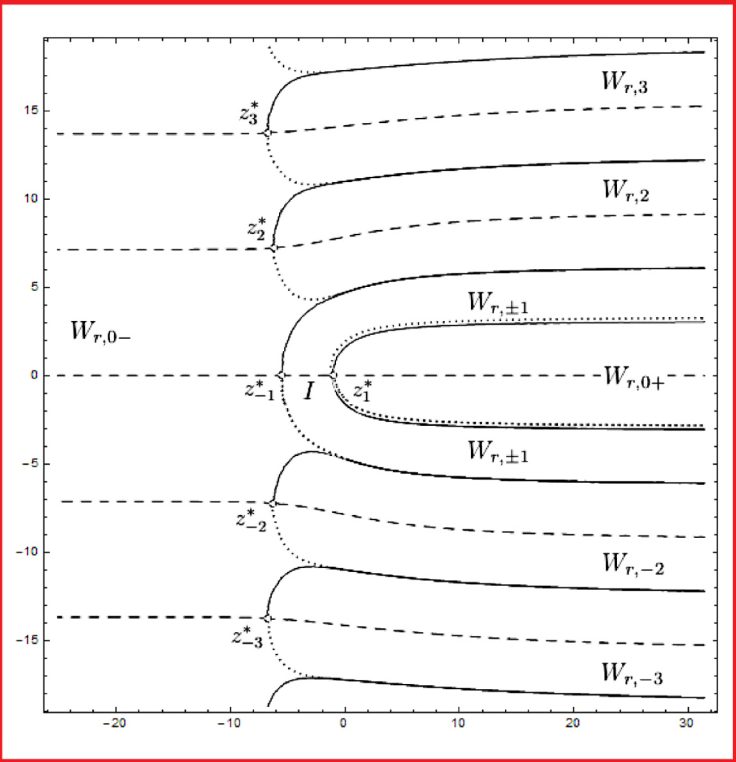
DISCRETE MATHEMATICS AND ITS APPLICATIONS

THE LAMBERT W FUNCTION

Its Generalizations and Applications



István Mező

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

The Lambert W Function

Discrete Mathematics and Its Applications
Series editors:
Miklos Bona, Donald L. Kreher, Douglas B. West

Algorithmics of Nonuniformity
Tools and Paradigms
Micha Hofri, Hosam Mahmoud

Handbook of Geometric Constraint Systems Principles
Edited by Meera Sitharam, Audrey St. John, Jessica Sidman

Introduction to Chemical Graph Theory
Stephan Wagner, Hua Wang

Extremal Finite Set Theory
Daniel Gerbner, Balazs Patkos

The Mathematics of Chip-Firing
Caroline J. Klivans

Computational Complexity of Counting and Sampling
Istvan Miklos

Volumetric Discrete Geometry
Karoly Bezdek, Zsolt Langi

The Art of Proving Binomial Identities
Michael Z. Spivey

Combinatorics and Number Theory of Counting Sequences
Istvan Mezo

Applied Mathematical Modeling
A Multidisciplinary Approach
Douglas R. Shier, K.T. Wallenius

Analytic Combinatorics
A Multidimensional Approach
Marni Mishna

50 years of Combinatorics, Graph Theory, and Computing
Edited By Fan Chung, Ron Graham, Frederick Hoffman, Ronald C. Mullin, Leslie Hogben, Douglas B. West

Fundamentals of Ramsey Theory
Aaron Robertson

Methods for the Summation of Series
Tian-Xiao He

The Lambert W Function
and its Generalizations and Applications
István Mező

Combinatorics of Permutations, Third Edition
Miklos Bona

https://www.routledge.com/Discrete-Mathematics-and-Its-Applications/book-series/CHDISMTHAPP

The Lambert W Function

Its Generalizations and Applications

István Mező

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

A CHAPMAN & HALL BOOK

First edition published 2022
by CRC Press
6000 Broken Sound Parkway NW, Suite 300, Boca Raton, FL 33487-2742

and by CRC Press
4 Park Square, Milton Park, Abingdon, Oxon, OX14 4RN

© 2022 István Mező

CRC Press is an imprint of Taylor & Francis Group, LLC

Reasonable efforts have been made to publish reliable data and information, but the author and publisher cannot assume responsibility for the validity of all materials or the consequences of their use. The authors and publishers have attempted to trace the copyright holders of all material reproduced in this publication and apologize to copyright holders if permission to publish in this form has not been obtained. If any copyright material has not been acknowledged please write and let us know so we may rectify in any future reprint.

Except as permitted under U.S. Copyright Law, no part of this book may be reprinted, reproduced, transmitted, or utilized in any form by any electronic, mechanical, or other means, now known or hereafter invented, including photocopying, microfilming, and recording, or in any information storage or retrieval system, without written permission from the publishers.

For permission to photocopy or use material electronically from this work, access www.copyright.com or contact the Copyright Clearance Center, Inc. (CCC), 222 Rosewood Drive, Danvers, MA 01923, 978-750-8400. For works that are not available on CCC please contact mpkbookspermissions@tandf.co.uk

Trademark notice: Product or corporate names may be trademarks or registered trademarks and are used only for identification and explanation without intent to infringe.

ISBN: 9780367766832 (hbk)
ISBN: 9781032223391 (pbk)
ISBN: 9781003168102 (ebk)

DOI: 10.1201/9781003168102

Typeset in CMR10 font
by KnowledgeWorks Global Ltd.

Publisher's note: This book has been prepared from camera-ready copy provided by the authors

Para mi querida esposa, és a kislányomak, Eszternek

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

Contents

Preface xiii

Historical prelude xv

About the author xix

Symbols xxi

I The classical Lambert W function 1

1 Basic properties of W 3

1.1 Equations solvable in terms of W . . . 3

1.1.1 Defining equation of W . . . 3

1.1.2 Equations of the form $xb^x = a$. The base b Lambert function . . . 5

1.1.3 Equations of the form $x + b^x = a$ . . . 6

1.1.4 Equations of the form $x + \log_b x = a$, and $x \cdot \log_b x = a$ 7

1.1.5 Equations of the form $x^x = a$, and $x^y = y^x$ . . . 9

1.2 Equations satisfied by W . . . 10

1.2.1 The exponential and logarithm of W . . . 10

1.2.2 W at some functional arguments . . . 11

1.2.3 Linear combination of two Lambert function values . . 11

1.3 Differential and integral formulas . . . 14

1.3.1 The derivative of W . . . 14

1.3.2 Higher order derivatives . . . 15

1.3.3 The generating function and the coefficients of the $p_n(x)$ polynomials . . . 17

1.3.4 Derivatives of $W(e^x)$ . . . 19

1.3.5 The differential equation of W . . . 20

1.3.6 Some integrals of W . . . 21

1.3.7 The Mellin transform of W . . . 23

1.3.8 The Laplace transform of $W(e^t)$ . . . 24

1.4 The Taylor series of W_0 . . . 25

1.5 The number of real solutions of xe^x. The two real branches of W . . . 26
1.6 Convexity and concavity of W_0 and W_{-1} . . . 29
1.7 The Ω constant . . . 30
1.7.1 Ω is transcendental . . . 31
1.7.2 The Adamchik integral (in a generalized form) . . . 32
1.7.3 A log-exponential integral . . . 34
1.7.4 A log-trigonometric integral . . . 37
Further notes . . . 38

2 The branch structure of the Lambert W function 39
2.1 The exponential function . . . 39
2.1.1 The mapping properties of exp . . . 39
2.2 The definition of the logarithm . . . 40
2.3 Branch cuts, covering maps, and Riemann surfaces . . . 43
2.3.1 The z-plane and the w-plane . . . 43
2.3.2 Branch cut and branch point . . . 43
2.4 Riemann surfaces . . . 45
2.4.1 Motivation . . . 45
2.4.2 Covering maps . . . 47
2.4.3 The notion of a surface . . . 47
2.4.4 The Riemann surface of the logarithm – heuristic way 48
2.4.5 The Riemann surface of the logarithm – exact way . . 50
2.4.6 The monodromy group of the Riemann surface of the logarithm . . . 53
2.5 The branches of the Lambert function . . . 54
2.5.1 The partition of the z-plane . . . 54
2.5.2 The branch separating curves . . . 54
2.5.3 The trisectrix of Hippias . . . 56
2.5.4 The closure of the branches . . . 59
2.5.5 The Riemann surface of the Lambert function . . . 63
2.5.6 Monodromies . . . 65
2.5.7 The fixed points of the Lambert function . . . 67
Further notes . . . 70

3 Unwinding number and branch differences 71
3.1 Why does the complex version of many identities fail? . . . 71
3.1.1 Non-identities regarding the logarithm . . . 71
3.1.2 Non-identities regarding roots . . . 72
3.2 The unwinding number . . . 73
3.2.1 The definition of the unwinding number . . . 73

3.2.2 The use of $\mathcal{K}(z)$ . . . 75
3.3 Corrected identities for the Lambert function . . . 76
3.3.1 Correcting (1.21) . . . 76
3.3.2 Correcting (1.23) . . . 78
3.4 The Wright ω function . . . 82
3.4.1 Looking at equation $x + \log x = a$ more carefully . . . 82
3.4.2 The definition and basic properties of ω . . . 83
3.4.3 The mapping properties of ω . . . 86
3.5 The branch difference function $\mathfrak{M}_{m,n}$ . . . 88
3.5.1 An equation for $\mathfrak{M}_{m,n}(z)$ . . . 88
3.5.2 An equation for $\mathfrak{M}_{0,-1}(z)$ . . . 90
Further notes . . . 93

4 Numerical approximations **95**
4.1 A concise series representation of W_k . . . 95
4.1.1 A first approximation of W . . . 95
4.1.2 The integral representation of $v(z)$ . . . 97
4.1.3 The $c_{k,m}$ constants . . . 99
4.1.4 The asymptotic series representation for W_k . . . 101
4.1.5 The analysis of the remainder term in the asymptotic series representation for W_k . . . 102
4.2 Series expansions and Padé approximants around $-1/e$ . . . 103
4.2.1 The behavior of W_{-1} and W_1 around $z = -1/e$ . . . 103
4.2.2 The notion of Padé approximant . . . 105
4.2.3 The Padé approximant of W_0 around zero . . . 107
4.3 Continued fraction representations for W_0 . . . 108
4.3.1 The notion of C-fractions . . . 108
4.3.2 Correspondence between C-fractions and Taylor series 109
4.3.3 The C-fraction expansion of $W_0(x)$ – via Hankel determinants . . . 110
4.3.4 The C-fraction expansion of $W_0(x)$ – via the *qd*-algorithm . . . 112
4.3.5 Associated continued fractions . . . 113
4.3.6 The analysis of the relative errors of C- and associated fractions . . . 116
4.3.7 The asymptotics of the elements in the C-fractions and associated fractions . . . 118
4.3.8 The limit of the elements in the C-fraction expansion of W_0 and Van Vleck's theorem . . . 120
Further notes . . . 121

II Generalized Lambert functions 122

5 Generalizations of the Lambert function 123
5.1 The generalized Lambert function . . . 123
5.1.1 The polynomial-exponential and rational-exponential type equations . . . 123
5.1.2 The definition of the generalized Lambert function . . 124
5.1.3 Simple cases . . . 125
5.2 Transformations of rational-exponential type equations . . . 126
5.3 The $(1,1)$-type function and the r-Lambert function . . . 127
5.4 A series representation for the $(1,1)$-type function . . . 128
Further notes . . . 131

6 The r-Lambert function 133
6.1 Equations solvable with the aid of W_r . . . 133
6.1.1 Some special values of W_r . . . 133
6.1.2 The base change formula . . . 135
6.1.3 An equation for the classical W function . . . 135
6.2 The derivative and integral of W_r . . . 136
6.2.1 The derivative of W_r . . . 137
6.2.2 Higher order derivatives . . . 137
6.2.3 Derivatives of $W_r(e^z)$ . . . 138
6.2.4 The derivative of W_r with respect to r . . . 139
6.2.5 The approximation of the r-Lambert function with the Lambert function . . . 139
6.2.6 The integral of W_r and of its powers . . . 140
6.3 The branch structure . . . 141
6.3.1 The real values . . . 141
6.3.2 Setting up the scene . . . 143
6.3.3 The branch separating curves in the z-plane . . . 144
6.3.3.1 The case when $r > 1/e^2$ . . . 145
6.3.3.2 The case when $r = 1/e^2$ . . . 151
6.3.3.3 The case when $0 < r < 1/e^2$ . . . 157
6.3.3.4 The case when $r < 0$ . . . 159
6.3.4 More on the points z_k^* . . . 162
Further notes . . . 166

III Applications 167

7 Physical applications 169
7.1 Falling body in resistant media . . . 169
7.1.1 Free fall in vacuum . . . 169
7.1.2 Taking the resistance of the media into account . . . 171

7.2 Paramagnets and magnetization . . . 173
7.2.1 The Brillouin function . . . 173
7.2.2 The inverse Langevin function . . . 174
7.3 Kepler's equation and Laplace's limit constant . . . 175
7.3.1 The Kepler equation and λ . . . 175
7.3.2 λ in terms of the $(1,1)$-Lambert function . . . 176
7.4 Thermodynamic properties of an ensemble in a quantum well 178
7.4.1 The quantum well . . . 178
7.4.2 Thermodynamics and ensembles in a quantum well . . 178
7.5 Einstein – Johnson sidewall correction in channel flow models 181
7.6 Radial speed and mass loss rate of solar wind . . . 182
7.6.1 The governing equation of the solar wind . . . 182
7.6.2 The mass loss rate due to solar wind . . . 184
7.7 Wien's displacement law . . . 186
7.8 Electric field around semi-infinite parallel plates . . . 187
7.9 The electric potential around a collision presheath in plasma 191
7.9.1 Sheath and presheath . . . 191
7.9.2 The potential χ in terms of W . . . 192
7.10 Delta-potential model for hydrogen molecule ion . . . 193
Further notes . . . 194

8 Biology, Ecology, and Probability 197
8.1 Cell proliferation . . . 197
8.1.1 The need for delay differential equations . . . 197
8.1.2 The Baker – Bocharov – Paul – Rihan model . . . 197
8.2 Random parasite and prey models . . . 199
8.2.1 The Random Parasite Equation . . . 199
8.2.2 The Random Predator Equation . . . 200
8.3 Blooming of phytoplanktons . . . 201
8.3.1 Sverdrup's approach . . . 201
8.3.2 Biomass growth and critical depth . . . 201
8.4 Enzyme kinetics . . . 204
8.4.1 The Michaelis – Menten equation . . . 204
8.4.2 Exact solution of the Michaelis – Menten equation . . . 204
8.4.3 Dark adaptation of the human eye . . . 205
8.5 The Lindley distribution . . . 205
8.5.1 The definition of the Lindley distribution . . . 205
8.5.2 Lindley and Lambert W . . . 206
8.6 The maximum of standard normal random variables . . . 207
8.6.1 Norming constants . . . 207
8.6.2 The norming constants and W . . . 207

8.7 Superstars and retweet graphs . . . 208
8.7.1 Tweets, retweets, and superstars . . . 208
8.7.2 The superstar model . . . 209
Further notes . . . 211

9 Mathematical applications 213
9.1 The infinite power tower . . . 213
9.1.1 Tetration . . . 213
9.1.2 The power tower in terms of W . . . 214
9.2 Counting trees . . . 214
9.2.1 Networks, minimal connectedness, and labelings . . . 214
9.2.2 Cayley's theorem, generating functions, and W . . . 216
9.3 The asymptotics of the Bell numbers . . . 217
9.3.1 Number of groupings – The Bell numbers . . . 217
9.3.2 The asymptotics of the Bell numbers with the aid of W 218
9.4 The associate space of an Orlicz space . . . 220
9.4.1 Basics on Orlicz spaces . . . 220
9.4.2 The Orlicz space dual of L_Φ . . . 221
9.5 Kapteyn series . . . 223
9.5.1 The notion of Kapteyn series . . . 223
9.5.2 The geometry of the Kapteyn domain and the r-Lambert function . . . 224
9.5.3 The proofs . . . 226
9.5.4 The visualization of p_a, q_a and the domain D_a . . . 227
9.6 The best constants for the estimate of Weierstrass canonical products . . . 229
9.7 Delay Differential Equations . . . 230
9.7.1 A simple example . . . 230
9.7.2 Some other examples . . . 232
Further notes . . . 233

Bibliography 235

Index 249

Preface

The Lambert W function is a relatively young mathematical function with plenty of interesting applications. The algebraic equations leading to the Lambert W function are known since centuries, but the first treatise on the W function itself did not appear until 1996. After the seminal paper of R. M. Corless et al. [39], scientists have found surprisingly many applications of this function in the theory of projectile motion, wave physics, cell growth models, solar wind, enzyme kinetics, plasma and quantum physics, general relativity, prey and parasite models, twitter events, and so on. During the writing of the text, I kept in mind all of those scientists who are not mathematicians, but would like to better understand the W function, or a given aspect of it which is particularly necessary in their work. Therefore the text is written to be accessible for everyone with only basic knowledge in calculus and complex numbers. This is particularly apparent in Chapter 2: when introducing notions like branch cut, or Riemann surface we give full details for the more elementary and much better known smaller cousin of W, the logarithm function. I hope that this is a valid explanation for my mathematician colleagues who would otherwise find the lengthier introductions and explanations unnecessary.

It turned out soon after 1996 that more general equations are there which come from practical applications but are not solvable with the help of the W function. Motivated by this revelation, several directions of generalization of this function arose in the last couple of decades. Many aspects of these generalizations are also incorporated in this text.

This book is the very first one in the English language[1] entirely dedicated to the Lambert W function, its generalizations, and its applications.

Our work intends to be a reference book which contains all the information one needs when trying to find a result on the Lambert function. In order to make the finding easier, the most important formulas and results are framed. But also, this is a book for an audience with wider interest: high school teachers can find motivation in Chapter 1 to examine equations not solvable by standard classroom methods; scientists from many fields may find interesting applications in the book; calculus professors can get ideas for teaching and practicing standard knowledge (derivation, integration, asymptotics, differential equations) on non-standard functions.

[1] There is an excellent book in Russian from the authors A. E. Dubinov, I. D. Dubinova, S. K. Saĭkov [50] on the Lambert function and its physical applications.

If a book targets a wider audience, it should be made clear what are the prerequisites. It depends on the given chapter. The first chapter needs basic real analysis, the second chapter requires some topology and complex function theory, and a very tiny bit of understanding group theory when the monodromy is discussed. The third chapter uses numerical approximations, matrix algebra, and computer programming. These cover the first part of the book. The second part does not need new knowledge, we discuss the generalized Lambert functions based on the tools we had developed in the first part. The third part is different: here we depart from mathematics, and discuss some applications of the W function and its generalizations. Here familiarity with Newtonian physics will be useful.

The researcher who is looking for interesting research problems can consult the *Further Notes* sections (especially in Chapter 2) in which questions are there for further research. The *Further Notes* sections contain not only research problems, but information for further studies on questions which, for some reason, are left out from this text. There are *Mathematica* codes on some pages of this book. Instead of typing these in, the reader can request them via e-mail: istvanmezo81@gmail.com.

In at least one aspect, however, the text is painfully incomplete. The generalizations of the Lambert function (the content of Part II) are just recently attracting attention from the researchers, and there are still many questions to work out; so, again, researchers looking for research problems, go ahead! There are many interesting questions here, and many papers to be written!

Certainly, it would have been impossible to put every application of W and its generalizations into this book. We selected the simpler and more accessible ones which might attract a wider audience. Such a selection is obviously subjective, but we hope that the reader will enjoy to learn about examples such as how the falling time of an object depends on the resistance of the air, or how the fuel consumption varies during the traveling of a rocket when the change in the mass caused by the same fuel consumption is taken into account.

From a pedagogical point of view, it is probably interesting to follow through the investigation of the properties of a function, which is arguably the simplest one outside the circle of the high school math class functions, such as log, sin and the others.

István Mező
Nanjing and Budapest, 2019–2021.

Historical prelude

Work of Lambert and Euler on what we call now the Lambert function

The history of the Lambert function began in 1758, when Johann Heinrich Lambert studied the trinomial equation[2]

$$x^m + px = q. \tag{0.1}$$

He found successive lower and upper estimate for a solution x in terms of q, p, and m, and noted that these lower and upper estimates converge to the same number – a solution of equation (0.1). The result reads as follows [108]:

$$x = \frac{q}{p} - \frac{q^m}{p^{m+1}} + m\frac{q^{2m-1}}{p^{2m+1}} - m\frac{3m-1}{2}\frac{q^{3m-2}}{p^{3m+1}} + \cdots .$$

(In the 1758 work of Lambert this was still not written out explicitly for general m.)

According to the online source of the collected works of Lambert [27], this 1758 treatise "contains some of the most important mathematical ideas Lambert ever conceived".

Lambert returns to equation (0.1) in 1770 and extends his results by finding series expansion for the positive integer powers of x [109, p. 229].

Leonhard Euler learned of Lambert's results in 1764 (the year when Euler invited Lambert to Berlin), and was quite excited, as Euler's letter to Goldbach testifies[3]. Euler studied the more general equation

$$x^\alpha - x^\beta = (\alpha - \beta)vx^{\alpha+\beta}, \tag{0.2}$$

and found a series expansion for the powers of a solution of this equation [57].

How does the Lambert W function come into the picture? Euler looked at special cases of (0.2), in particular, when $\alpha = \beta$. To see what happens in this case, divide both sides of (0.2) by $\alpha - \beta$, and let β tend to α:

$$\lim_{\beta\to\alpha} \frac{x^\alpha - x^\beta}{\alpha - \beta} = \lim_{\beta\to\alpha} vx^{\alpha+\beta}.$$

[2] An equation consisting of three monomial terms.

[3] See the end of [39] for historical notes on this question.

The limit on the left-hand side is $x^\alpha \log x$, and on the right-hand side we clearly have $vx^{2\alpha}$. That is, the equation transforms into

$$\log x = vx^\alpha. \tag{0.3}$$

Now multiply this with α, and use the good old identity $\alpha \log x = \log(x^\alpha)$ to get

$$\log x^\alpha = \alpha v x^\alpha.$$

Substitute $y = x^\alpha$, and set $u = \alpha v$, so that

$$\log y = uy. \tag{0.4}$$

This is just equation (0.3) with $\alpha = 1$. We therefore get – what Euler also noted – that it is enough to solve (0.4) if we want to solve (0.2) (in the case when $\alpha = \beta$).

Equation (0.4) directly leads to the usual definition of the Lambert W function. Indeed, take the exponential of both sides, and rearrange:

$$ye^{-uy} = 1.$$

Substitute $z = -uy$ (so that $y = -z/u$):

$$-\frac{z}{u}e^z = 1,$$

or

$$ze^z = -u.$$

In place of $-u$ we just write a:

$$ze^z = a. \tag{0.5}$$

The solution of this equation is denoted by $W(a)$, and this is how we define the Lambert W function. There are, in fact, infinitely many solutions to (0.5) – this will be studied in great detail later.

Why Lambert W function and not Euler W function?

Above is the arc from the investigations of Lambert to the emergence of the W function which now bears his name. The reader might have noticed that Lambert was not really the one who got the closest to the defining equation of W, but Euler. Brian Hayes [79] put this question even more harshly:

> Lambert, who wrote on everything from cartography to photometry to philosophy, never published a word on the function that now bears his name. It was his eminent colleague Leonhard Euler who first described a variant of the W function in a paper published in 1779, two years after Lambert's death. So why isn't it called the Euler W function?

Hayes gives not one, but two, answers. The first answer is that Euler gave credit to Lambert for the earliest work on the subject. In fact, Euler highly respected the contributions of Lambert. In the above mentioned 1779 paper, Euler calls him "acutissimi ingenii Lambertus" which may be freely translated as the "ingenious engineer Lambert". (This was noted at the end of [39].)

The other answer of Hayes cites Robert M. Corless, David J. Jeffrey, and Donald E. Knuth saying "naming yet another function after Euler would not be useful." This consideration led to the proposal that W should be named after J. H. Lambert.

Corless and his colleagues wrote a series of influential papers on W in the 90s, and introduced W to the wider scientific community. Since then, hundreds of papers were born on the mathematical properties of W and its applications in a surprisingly wide variety of areas in other sciences.

Why W?

After we have learned why the Lambert function is called so, one thing still remains to be answered: why do we use the letter W, and why not, for example, the probably more reasonable L? Hayes dedicated a whole article to this question [79]. We can learn from his work that F. N. Fritsch, R. E. Schafer, and W. P. Crowley wrote a publication in which they used the letter w in the defining equation $we^w = x$ [62]. Gaston H. Gonnet wrote the routine for the *Maple* mathematical software to calculate the solutions of the equation $we^w = x$. In this software, the common functions are capitalized, so Gonnet used the letter W instead of w.

There is another reason – noted by Corless et al. [39] – why the letter W is appropriate: many aspects of the W function were investigated by the eminent mathematician E. M. Wright between 1949 and 1977 [185, 186, 187, 188].

The letter "w" also appears in the 1925 book of G. Pólya and G. Szegő [144] in connection with the defining equation of the Lambert function. Problem 209 asks for the expansion of z in terms of w if the two numbers are connected via the equation $ze^{-z} = w$. We see that $z = -W(-w)$. The authors refer to the original works of Euler when discussing the solution (which is just the Taylor series (1.52), up to signs).

[4]The bibliography was taken from http://www-history.mcs.st-and.ac.uk/Biographies/Lambert.html

The life of Johann Heinrich Lambert (26 August 1728 - 25 September 1777)[4]

The Lambert family was originally from Lorraine which was an autonomous territory at that time (in 1766, Lorraine was annexed by the Kingdom of France). By religious reasons, the Lambert family moved to the Republic of Mulhouse in 1635. Mulhouse was part of an association of ten free imperial cities in Alsace, but in 4 January 1798 it became a part of France after a vote by its citizens.

Heinrich Lambert was born into a big family, he had four brothers and two sisters. His father, Lukas Lambert, as well as his grandfather were tailors. Lukas Lambert got married in 1724 with Elizabeth Schmerber. The family had no sufficient income to support and educate their children, and thus Heinrich left the school at age twelve to help his father. He did not finish his studies, however. In his spare time in the evenings, the young Heinrich studied science on his own. From the age of fifteen, he worked as clerk, private tutor, and secretary. From 1748 he became gradually known by the scientific community, and was elected to the Literary Society of Chur (the city where he lived at that time) and to the Swiss Scientific Society based in Basel. Lambert published his first article in 1755 on caloric heat. In the subsequent years (1758-1761), he published three books. The first one was on the passage of light through various media, the second on photometry, while the third was on cosmology. In this latter book Lambert made the statement that the Universe is composed of galaxies which are composed of stars.

In 1764 Lambert was invited to the Prussian Academy of Sciences in Berlin by Euler. At first, Frederick II refused to accept Lambert by his eccentric behaviour and unusual appearance (Lambert consciously refused to conform with the conventions of the upper classes). Once Frederick II got to know Lambert, he acknowledged him.

In 1766 Lambert wrote a study on the parallel postulate. By assuming that it is false, he discovered a number of non-Euclidean theorems. Two years later he presented his theorem on the irrationality of π. He proved that if x is rational, then $\tan x$ is irrational. Since $\tan \frac{\pi}{4} = 1$, a rational number, π cannot be rational.

About the author

István Mező is a Hungarian mathematician, born in Debrecen. He obtained his PhD in 2010 at the University of Debrecen. He worked at this institute until 2014. After two years of Prometeo Professorship at the Escuela Politécnica Nacional (Quito, Ecuador) between 2012 and 2014, he moved to Nanjing, China where he is now a full-time research professor. Since 2021, he is also a professor at Óbuda University, Budapest, Hungary.

Dr. Mező is currently working on Combinatorics and Special Function Theory.

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

Symbols

Here we collect some standard notations which will be used through the book, and also the notations introduced through the course of our endeavor in the world of the Lambert W function.

Symbol Description

$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{C}^*$	set of non-zero complex numbers
$\mathbb{Z}$	set of integer numbers
$\mathbb{Z}_{\setminus 0}$	set of non-zero integers
$f(A)$	image of $A \subset X$ by the function $f : X \to Y$: $f(A) = \{y \in Y : \exists x \in A \text{ such that } y = f(x)\}$
$\mathrm{dom}(f)$	domain of the function f
$\mathrm{ran}(f)$	range of the function $f : X \to Y$; $\mathrm{ran}(f) = f(X)$
∂A	topological boundary of the set A
Ω	omega constant: $\Omega = W_0(1)$
$\mathcal{K}$	unwinding number: $\mathcal{K}(z) = \left\lceil \frac{\Im(z)-\pi}{2\pi} \right\rceil$
ω	Wright omega function: $\omega(z) = W_{\mathcal{K}(z)}(e^z)$
$\mathfrak{M}$	branch difference: $\mathfrak{M}_{m,n}(z) = W_m(z) - W_n(z)$
$(\mathcal{M}f)(s)$	Mellin transform of the function f at the point s
$(\mathcal{L}f)(s)$	Laplace transform of the function f at the point s
$[L/M]_f$	Padé approximant of f with order L and M
$\mathrm{sgn}(x)$	signum of x: $\mathrm{sgn}(x)$ is negative for negative x, positive for positive x, and $\mathrm{sgn}(0) = 0$
$\Re(z)$	real part of the complex number z
$\Im(z)$	imaginary part of the complex number z
$[k]$	Unit of the physical quantity k. For example: $[m]$ =kg, where m denotes the mass of an object
$a_n \sim b_n$	Asymptotic equivalence of the sequences (a_n) and (b_n): $\lim_{n\to\infty} a_n/b_n = 1$

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

Part I

The classical Lambert W function

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

1

Basic properties of W

In this chapter we study those properties of the Lambert W function which do not need knowledge on the complex function theoretic behavior of W.

We study in detail the equations which are solvable in terms of W, give a general formula for the derivatives, determine some integrals involving W, rigorously check the domain and range of the real Lambert function, and present convexity-concavity properties. At the end of the chapter we study the constant Ω, which is a special value of W.

1.1 Equations solvable in terms of W

1.1.1 Defining equation of W

The Lambert W function is defined as the solution of the equation

$$xe^x = a, \tag{1.1}$$

where a is the known parameter, and e is the base of the natural logarithm. We look for the unknown x which satisfies (1.1). This x is the solution, so we write $x = W(a)$. We call (1.1) as the *defining equation of* W.

We can therefore equivalently say that $W(a)$ is the function which satisfies the equation

$$\boxed{W(a)e^{W(a)} = a.} \tag{1.2}$$

There is another equivalent way we may define W. Putting $f(x) = xe^x$, we have that

$$f(x) = a. \tag{1.3}$$

The inverse of a function f is the function, denoted by f^{-1}, for which

$$f(f^{-1}(x)) = x, \quad \text{and} \quad f^{-1}(f(x)) = x.$$

So, if we want to solve (1.3) by finding x, we may use the inverse of f to write

$$f^{-1}(f(x)) = f^{-1}(a).$$

DOI: 10.1201/9781003168102-1

On the left-hand-side we have x, so the solution is given as:

$$x = f^{-1}(a).$$

Since $W(a) = x = f^{-1}(a)$, we have that W is the *inverse function* of xe^x. This observation will be pushed further in Section 1.5.

Some special values of the W function

Some special values of W can immediately be deduced. If $a = 0$ in (1.1), then $x = 0$ must hold, since the exponential function is never zero. Thus the solution of (1.1) when $a = 0$ is $x = 0$. Hence

$$\boxed{W(0) = 0.}$$

If, in turn, $a = e$, we get that $x = 1$ is a solution. Indeed,

$$1 \cdot e^1 = e.$$

This means that

$$\boxed{W(e) = 1.} \tag{1.4}$$

Similarly, if $a = -\frac{1}{e}$, then $x = -1$ solves (1.1):

$$(-1)e^{-1} = -\frac{1}{e} = a,$$

whence

$$\boxed{W\left(-\frac{1}{e}\right) = -1} \tag{1.5}$$

follows.

Another trivial, but still nicely looking, special case is

$$\boxed{W\left(e \cdot e^e\right) = e,} \tag{1.6}$$

which is just the consequence of the fact that W is the inverse of xe^x.

Equation (1.1) seems to be too special – but only at first sight. There are plenty of equations that can be transformed into the form of (1.1) and thus can be solved with the aid of W. In the following subsections we study such equations.

That (1.1) seems to be too "particular" still does not mean that it is not of high importance. B. Hayes writes in his paper [79]: *"Those for whom W is a favorite function want to see it elevated to the canon of standard textbook functions, alongside log and sine and square root. [...] The advocates of W do make a strong case. In a 2002 paper, Corless and Jeffrey argue that W is in some sense the smallest step beyond the present set of elementary functions. 'The Lambert W function is the simplest example of the root of an exponential polynomial; and exponential polynomials are the next simplest class of functions after polynomials'."* The Corless-Jeffrey paper in question is [43].

1.1.2 Equations of the form $xb^x = a$. The base b Lambert function

We know from our high school studies that the exponential function can be defined with many bases: 3^x, $\left(\frac{1}{2}\right)^x$, etc. To solve equations like $3^x = 4$, we need logarithm of base three: $x = \log_3(4)$. But we also know that

$$\log_a(b) = \frac{\log_c(b)}{\log_c(a)} \quad (a, b, c > 0;\, a, c \neq 1). \tag{1.7}$$

(The natural base logarithm is denoted by log here and elsewhere.) This identity tells us that whatever is the base we want to calculate the logarithm for (in our case the base is three), we can always express that in terms of another base; knowing the logarithm in one base is enough. Applying (1.7), we express $\log_3(4)$ in base e logarithm:

$$\log_3(4) = \frac{\log(4)}{\log(3)}.$$

A similar fact is true for the W function. Let us suppose that the solution of

$$xb^x = a \tag{1.8}$$

is given by the "base b" Lambert function, let us denote it by $W_b(a)$. Our aim is to express $W_b(a)$ in terms of the base e function which is just our $W(a)$.

First, note that $b^x = e^{x \log b}$. We now keep everything real. Complex solutions will be thoroughly studied later; we therefore suppose that $b > 0$ (but $b \neq 1$) so that the logarithm stays real. By this expression for b^x, (1.8) is the same as

$$xe^{x \log b} = a.$$

Now substitute $y = x \log b$:

$$\frac{y}{\log b} e^y = a,$$

or

$$ye^y = a \log b.$$

This is soluble in terms of the classical (base e) Lambert function:

$$y = W(a \log b).$$

Since $x = \frac{y}{\log b}$, and $x = W_b(a)$ as well, we arrive at the *base-change formula*

$$\boxed{W_b(a) = \frac{W(a \log b)}{\log b} \quad (0 < b \neq 1).} \tag{1.9}$$

Our claim has been proven: it is enough to know W_b in base $b = e$.

1.1.3 Equations of the form $x + b^x = a$

The Lambert function is capable to solve not only $xe^x = a$, but also its "additive variant": $x + e^x = a$. Let us put this latter equation into a more general form, with an arbitrary base:

$$x + b^x = a. \tag{1.10}$$

Here, again, b is positive, and different from one. How to solve this equation? Let us take the exponential of both sides. Since we have base b, the base b exponential is more convenient:

$$b^{x+b^x} = b^a.$$

We use the identity $b^{x+y} = b^x b^y$:

$$b^x b^{b^x} = b^a.$$

If we substitute $y = b^x$, we immediately see that we have our familiar equation

$$yb^y = b^a.$$

The solution of this equation is given by the base b Lambert function:

$$y = W_b(b^a).$$

We look for x, and we know that $x = \log_b y$, and thus

$$x = \log_b(W_b(b^a)).$$

In Section 1.2 we shall see that the logarithm of the Lambert function (in whatever base) can always be expressed in terms of W itself. To see this, we refer to (1.21); where the base b version comes by simply changing the base of the logarithms and of W: $\log_b W_b(a) = \log_b a - W_b(a)$. It is therefore possible to get rid of $\log_b$ in the last displayed formula. Applying (1.21) we get our solution to (1.10):

$$\boxed{x + b^x = a \quad \Longrightarrow \quad x = a - W_b(b^a).} \tag{1.11}$$

In terms of the base e Lambert function, this reads as

$$\boxed{x + b^x = a \quad \Longrightarrow \quad x = a - \frac{W(b^a \log b)}{\log b}.} \tag{1.12}$$

Example: $3x + 5^{2x} = 7$

Let us apply our results and solve the equation

$$3x + 5^{2x} = 7.$$

We substitute $y = \frac{3}{x}$ to get rid of the factor of the linear term:

$$y + 5^{\frac{2}{3}y} = 7,$$

or, which is the same,

$$y + \left(5^{\frac{2}{3}}\right)^{y} = 7.$$

This equation can be solved with the help of (1.11). We find that $a = 7$, $b = 5^{\frac{2}{3}}$, and

$$y = 7 - W_{5^{\frac{2}{3}}}\left(5^{\frac{14}{3}}\right),$$

and x, the solution of the original equation, is one-third of y:

$$x = \frac{7}{3} - \frac{1}{3}W_{5^{\frac{2}{3}}}\left(5^{\frac{14}{3}}\right).$$

If we rather want to express this solution in terms of the base e Lambert function, we use (1.12) instead:

$$x = \frac{7}{3} - \frac{W\left(\frac{2}{3}5^{\frac{14}{3}}\log 5\right)}{2\log 5},$$

In Chapter 4 we shall see how to evaluate W numerically, so that one can find the numerical value of the above x by computer. One gets that

$$x \approx 0.525294.$$

1.1.4 Equations of the form $x + \log_b x = a$, and $x \cdot \log_b x = a$

Taking the base b logarithm of both sides of $xb^x = a$, we get the equivalent equation

$$\log_b x + x = \log_b a,$$

and its solution is the same as that of $xe^x = a$:

$$\log_b x + x = \log_b a \Longrightarrow x = W_b(a).$$

The constant is arbitrary on the right-hand side; we therefore write

$$\boxed{x + \log_b x = a \Longrightarrow x = W_b(b^a).} \tag{1.13}$$

This can be generalized to a much more general equation:

$$\boxed{x^c + d\log_b\left(hx^f\right) = a \Longrightarrow x = \left[\frac{df}{c}W_b\left(\frac{c}{df}\left(\frac{b^{\frac{a}{d}}}{h}\right)^{\frac{c}{f}}\right)\right]^{\frac{1}{c}}.} \tag{1.14}$$

The calculations are left as an exercise.

The "multiplicative version"

$$x \cdot \log_b x = a$$

can be solved by substituting $x = b^y$:

$$b^y \cdot \log_b b^y = a,$$

or

$$yb^y = a.$$

This equation is just the defining equation of the base b Lambert function, so

$$y = W_b(a).$$

Recall that $x = b^y$, whence

$$x \cdot \log_b x = a \Longrightarrow x = b^{W_b(a)}.$$

We can simplify this expression by noticing that for the base b Lambert function we have $W_b(a)b^{W_b(a)} = a$, and thus

$$b^{W_b(a)} = \frac{a}{W_b(a)}.$$

The previous displayed equation can therefore be rewritten as

$$\boxed{x \cdot \log_b x = a \Longrightarrow x = \frac{a}{W_b(a)}.} \tag{1.15}$$

Sometimes the following generalization can be useful:

$$\boxed{x^a \cdot \log_b(cx^d) = a \Longrightarrow x = \left[\frac{d}{a^2} W_b\left(\frac{a^2}{d} c^{\frac{a}{d}}\right)\right]^{-\frac{1}{a}}.} \tag{1.16}$$

Example: $x \cdot \log_2(5x) = 3$

To solve $x \cdot \log_2(5x) = 3$, we substitute $y = 5x$:

$$\frac{1}{5} y \log_2(y) = 3,$$

or

$$y \log_2(y) = 15.$$

By (1.15) we calculate that

$$y = \frac{15}{W_2(15)},$$

so

$$x = \frac{3}{W_2(15)}.$$

In terms of the classical Lambert function this solution has the form

$$x = \frac{3\log 2}{W(15\log 2)},$$

where we used the base change formula (1.9). The numerical value of x is 1.17459.

1.1.5 Equations of the form $x^x = a$, and $x^y = y^x$

Notice that (1.15) helps to solve the equation

$$x^x = a.$$

Just take logarithm:

$$x^x = a \Longrightarrow x \cdot \log x = \log a,$$

thus, according to (1.15), $x = \log a / W(\log a)$.

$$\boxed{x^x = a \Longrightarrow x = \frac{\log a}{W(\log a)}.} \tag{1.17}$$

Now let us try to solve

$$x^y = y^x.$$

Transforming both sides with the use of $x^y = e^{y\log x}$, we get

$$e^{y\log x} = e^{x\log y}. \tag{1.18}$$

If we restrict ourselves to x and y such that the exponents are real (i.e., $x, y > 0$), then the equality of (1.18) is equivalent to the equality of the exponents:

$$y\log x = x\log y.$$

Let us separate the variables:

$$\frac{1}{x}\log x = \frac{1}{y}\log y.$$

Substituting $z = 1/x$, we get that

$$z\log\frac{1}{z} = \frac{1}{y}\log y, \quad \text{or} \quad z\log z = -\frac{1}{y}\log y.$$

This equation is again solvable by (1.15):

$$z = \frac{-\frac{1}{y}\log y}{W\left(-\frac{1}{y}\log y\right)}.$$

For x, this means that

$$x = -\frac{y}{\log y} W\left(-\frac{1}{y}\log y\right).$$

Let us write our result in its final form:

$$\boxed{x^y = y^x \Longrightarrow x = -\frac{y}{\log y} W\left(-\frac{1}{y}\log y\right).} \tag{1.19}$$

1.2 Equations satisfied by W

We defined W through an equation. The unknowns of equations are usually denoted by x, y, and the known parameters are denoted by the letters a, b, c. Just as we did in (1.1) and in the subsequent equations, this is why we used the notation $W(a)$. However, we are used to denoting arguments of functions by x, y, z, and not with a. From now on, we will concentrate on the functional behavior of the Lambert function, so we will write $W(x)$ in place of $W(a)$.

Another remark is due. All the base b Lambert functions can be expressed in terms of the classical (base e) Lambert function, and we therefore will study only the latter without losing any generality. We spare a lot of space and time by not writing out explicitly the two redundant statements for both W_b and $W = W_e$.

1.2.1 The exponential and logarithm of W

During our high school studies, we often apply rules to solve (exponential, logarithmic, trigonometric) equations. These "rules" are called, in math jargon, functional equations. We know many of these: $a^{x+y} = a^x \cdot a^y$, $\log(xy) = \log x + \log y$, and so on.

The Lambert function also satisfies interesting functional equations; the most straightforward are the following. From (1.2) (with the renaming $a \to x$) we immediately get that the exponential of the W function is expressible in a very simple way by W:

$$\boxed{e^{W(x)} = \frac{x}{W(x)}.} \tag{1.20}$$

By taking the logarithm of both sides of (1.2), we infer that the logarithm

of W is an easy expression of W itself:

$$\log W(x) + W(x) = \log x.$$

That is,

$$\boxed{\log W(x) = \log x - W(x) \quad (x > 0).} \tag{1.21}$$

(If $x > 0$, all the expressions are positive, so there is no problem of taking logarithms.)

1.2.2 W at some functional arguments

There is still more in (1.2) to be discovered. We shall prove that

$$\boxed{W(x \cdot \log x) = \log x \quad \left(x \geq \frac{1}{e}\right).} \tag{1.22}$$

We make the substitution $x \to x \cdot \log x$ in (1.2):

$$W(x \cdot \log x) e^{W(x \cdot \log x)} = x \cdot \log x.$$

For the easier formulation, let us denote $W(x \cdot \log x)$ by y:

$$y e^y = x \cdot \log x.$$

It is now easy to see that if e^y is chosen to be x, then $y = \log x$, and the equation is satisfied. We therefore have that

$$y = W(x \cdot \log x) = \log x,$$

as we stated.

Why this relation holds only for $x \geq \frac{1}{e}$ will be clear later: we shall see that for $x < \frac{1}{e}$ the $W(x \log x)$ function becomes complex, while log is still real (as far as $x > 0$), so the two sides cannot be equal.

Notice that by substituting respectively $x = 1, e, -\frac{1}{e}$, we recover the special values of W given in Section 1.1.1.

Substituting $x \to \frac{1}{x}$ in (1.22), we get

$$\boxed{W\left(-\frac{\log x}{x}\right) = -\log x \quad (0 \leq x \leq e).}$$

1.2.3 Linear combination of two Lambert function values

In this subsection we first prove the validity of the functional equation

$$\boxed{aW(x) + bW(y) = W\left(\left(\frac{x}{W(x)}\right)^a \left(\frac{y}{W(y)}\right)^b (aW(x) + bW(y))\right),} \tag{1.23}$$

and then deduce some simple consequences of it.

We look for an expression for $aW(x) + bW(y)$, so we write this into the defining equation (1.2):

$$[aW(x) + bW(y)]e^{aW(x)+bW(y)} = f(x, y, a, b).$$

Here on the right-hand side we have a function of x, y, a, and b. The solution of this equation is, by definition, $W(f(x, y, a, b))$. That is,

$$W(f(x, y, a, b)) = aW(x) + bW(y). \tag{1.24}$$

We rewrite $f(x, y, a, b)$ as

$$f(x, y, a, b) = [aW(x) + bW(y)]e^{aW(x)+bW(y)} =$$

$$[aW(x) + bW(y)]e^{aW(x)}e^{bW(y)}.$$

Recall that $e^{W(x)} = x/W(x)$ (see (1.20)). Upon elevating this to the ath and bth power, we get that

$$e^{aW(x)} = \left(\frac{x}{W(x)}\right)^a; \quad e^{bW(y)} = \left(\frac{y}{W(y)}\right)^b,$$

and these are substituted into the last expression of $f(x, y, a, b)$. We get that

$$f(x, y, a, b) = [aW(x) + bW(y)] \left(\frac{x}{W(x)}\right)^a \left(\frac{y}{W(y)}\right)^b.$$

Now we use this expression and (1.24) to arrive at (1.23).

Two immediate consequences are given by specializing (1.23) with $a = b = 1$, and $a = 1$, $b = -1$, respectively:

$$\boxed{\begin{aligned} W(x) + W(y) &= W\left(\frac{xy}{W(x)} + \frac{xy}{W(y)}\right), \\ W(x) - W(y) &= W\left(\frac{x}{W(x)}\frac{W(y)}{y}(W(x) - W(y))\right). \end{aligned}} \tag{1.25}$$

Some consequences of the linear combination formulas

We shall see some special Lambert W values which can be deduced with the aid of the linear combination formula, and (1.25), in particular. The W function at $x = 1$ is a special number which has its independent interest in several branches of mathematics [58]. The $W(1)$ constant is denoted by Ω:

$$\boxed{\Omega := W(1).} \tag{1.26}$$

Notice that, since $W(1)e^{W(1)} = 1$, Ω satisfies the equation

$$\boxed{\Omega e^{\Omega} = 1.}$$

Taking logarithm of both sides, we get yet another equation

$$\boxed{\Omega + \log \Omega = 0, \quad \text{or} \quad \log \frac{1}{\Omega} = \Omega.}$$

A different expression for this constant is given upon letting $x = y = 1$ in (1.25):

$$\boxed{W\left(\frac{2}{\Omega}\right) = 2\Omega.} \tag{1.27}$$

Next, let $x = e$ and $y = 1$. It easily comes that

$$\boxed{W\left(e + \frac{e}{\Omega}\right) = 1 + \Omega.}$$

Further results come when $x = 1$ and $y = ze^z$. Since $W(ze^z) = z$, we get from (1.25) that

$$\boxed{\Omega + z = W\left(\frac{ze^z}{\Omega} + \frac{ze^z}{z}\right) = W\left(e^z(ze^{\Omega} + 1)\right).} \tag{1.28}$$

In particular (by setting $z = 2$),

$$W\left(\frac{2e^2}{\Omega} + e^2\right) = 2 + \Omega.$$

Using the identity $\frac{1}{\Omega} = e^{\Omega}$, our expression can also be written as $W(2e^{\Omega+2} + e^2) = 2 + \Omega$.

A neat formula can be found for the real multiples of Ω also:

$$\boxed{\Omega x = W\left(\frac{x}{\Omega^{x-1}}\right) \quad \left(x \geq -\frac{1}{e}\right).} \tag{1.29}$$

The proof is done by easy symbolic manipulation. The statement holds if and only if

$$\Omega x e^{\Omega x} = \frac{x}{\Omega^{x-1}}.$$

This is equivalent to

$$e^{\Omega x} = \frac{1}{\Omega^x}, \tag{1.30}$$

or

$$\Omega^x e^{\Omega x} = 1.$$

The left-hand side is just $(\Omega e^{\Omega})^x$. The base, Ωe^{Ω} is one by definition, and its xth power is therefore also one, being equal to the right-hand side.

We still have to show that our identity is valid only when $x \geq -1/e$. The argument of W cannot be smaller than $-1/e$, because below this point W becomes complex, while Ωx remains real for all real x. It therefore comes that $\frac{x}{\Omega^{x-1}} \geq -1/e$ is necessary. An inspection of the properties of the function $\frac{x}{\Omega^{x-1}}$ shows that it is increasing, thus the limiting x we are looking for is the one for which

$$\frac{x}{\Omega^{x-1}} = -1/e.$$

This can be solved by the Lambert function, and the solution is $-1/e$.

A nice particular case of (1.29) (which comes by setting $x = e$) is:

$$\boxed{W\left(\frac{e\Omega}{\Omega^e}\right) = e\Omega.}$$

Talking about formulas for $W(x) + W(y)$, we remark that if this sum is known, then we can determine $W(x)W(y)$ easily:

$$\boxed{W(x) + W(y) = a \Longrightarrow W(x)W(y) = e^{-a}xy.} \tag{1.31}$$

Indeed, rearranging the right-hand side, we have

$$e^a = \frac{x}{W(x)}\frac{y}{W(y)} = e^{W(x)}e^{W(y)} = e^{W(x)+W(y)},$$

and this is equivalent to the left-hand side of (1.31).

To close this section, we remark that the case when $x = 1/\Omega$ in (1.30) yields the quite handsome identity

$$\boxed{\Omega^{\frac{1}{\Omega}} = \frac{1}{e}.}$$

(Note that this, however, is a simple consequence of the identity $\Omega e^{\Omega} = 1$, elevated to the $(1/\Omega)$-th power.)

1.3 Differential and integral formulas

1.3.1 The derivative of W

When studying a function, we certainly ask about the derivative and primitive function of it. The derivative of $W(x)$ is easy to find. Take the derivative of (1.2) (but now we use x in place of a) on both sides:

$$W'(x)e^{W(x)} + W(x)W'(x)e^{W(x)} = 1.$$

After a rearrangement, we have an expression for $W'(x)$:

$$W'(x) = \frac{1}{e^{W(x)}(1+W(x))}.$$

In place of $e^{W(x)}$ we write $x/W(x)$, and we get that

$$\boxed{W'(x) = \frac{W(x)}{x(1+W(x))}.} \tag{1.32}$$

The derivative of $W(x)$ becomes singular when $W(x) = -1$. This occurs when $x = -\frac{1}{e}$ (recall (1.5)). At this point of singularity interesting things happen; we will investigate these questions later, when we study the branch structure of the Lambert function.

At the point $x = 0$ the derivative will not be singular. The study of the Taylor series of $W(x)$ around zero[1] will reveal that $\lim_{x\to 0} \frac{W(x)}{x} = 1$ (see (1.53)). Therefore,

$$\boxed{W'(0) = 1.}$$

For $x = 1$ and remembering that $W(1) = \Omega$:

$$\boxed{W'(1) = \frac{\Omega}{1+\Omega}.}$$

1.3.2 Higher order derivatives

It is possible to find the general formula for the nth derivative of $W(x)$:

$$\boxed{W^{(n)}(x) = \frac{W^n(x)p_n(W(x))}{x^n(1+W(x))^{2n-1}} \quad (n \geq 1).} \tag{1.33}$$

Here p_n is a polynomial which satisfies the recursion

$$p_{n+1}(x) = (1+x)p_n'(x) - (nx + 3n - 1)p_n(x), \tag{1.34}$$

with the initial value

$$p_1(x) = 1.$$

When writing $p_n(W(x))$ we mean that at every occurrence of x one must substitute $W(x)$. For example, by the recursion

$$p_2(x) = (1+x)\cdot 0 - (x+3-1)\cdot 1 = -x - 2.$$

Thus

$$\boxed{W''(x) = \frac{W^2(x)(-W(x)-2)}{x^2(1+W(x))^3}.}$$

[1] To be more precise, we are talking about the Taylor series of the *principal branch* of

Let us prove (1.33) and (1.34). The derivative given in (1.32) shows that $p_1(x) = 1$, indeed. Now we suppose that (1.33) holds, and we try to write $W^{(n+1)}(x)$ in the form

$$\frac{W^{n+1}(x)p_{n+1}(W(x))}{x^{n+1}(1+W(x))^{2(n+1)-1}}. \tag{1.35}$$

It will turn out that this is possible only if $p_n(x)$ satisfies (1.34). To shorten the subsequent lengthy formulas, we do not write out the argument of p_n and W. Taking the derivative of (1.33), and simplifying a bit, we get

$$W^{(n+1)} = \frac{W^{n-1}}{x^{n+1}(1+W)^{2n}}\left[(np_nW' + p_n'WW')x(1+W) - \right.$$
$$\left. Wp_n(n(1+W) + x(2n-1)W')\right].$$

Let us write this expression such that it resembles more (1.35):

$$W^{(n+1)} = \frac{W^{n+1}}{x^{n+1}(1+W)^{2n+1}}\frac{1+W}{W^2}[\cdots].$$

Clearly, the part $\frac{1+W}{W^2}[\cdots]$ must be p_{n+1}. That is,

$$p_{n+1} = \frac{1+W}{W^2}\left[(np_nW' + p_n'WW')x(1+W) - \right.$$
$$\left. Wp_n(n(1+W) + x(2n-1)W')\right].$$

To make further simplifications, let us use (1.32) inside the square brackets. We get

$$p_{n+1} = \frac{1+W}{W^2}\left[np_nW + p_n'W^2 - Wp_nn(1+W) - \frac{W^2}{1+W}p_n(2n-1)\right] =$$
$$\frac{1+W}{W^2}\left[p_n'W^2 - W^2p_nn - \frac{W^2}{1+W}p_n(2n-1)\right] =$$
$$(1+W)p_n' - (1+W)p_nn - p_n(2n-1) = (1+W)p_n' - (nW+3n-1)p_n.$$

We see that (1.34) indeed holds, if we consider p_{n+1} as a polynomial of W.

For possible applications, we list here the first p_n polynomials.

$p_1(x) = 1,$

$p_2(x) = -x - 2,$

$p_3(x) = 2x^2 + 8x + 9,$

$p_4(x) = -6x^3 - 36x^2 - 79x - 64,$

$p_5(x) = 24x^4 + 192x^3 + 622x^2 + 974x + 625,$

$p_6(x) = -120x^5 - 1\,200x^4 - 5\,126x^3 - 11\,758x^2 - 14\,543x - 7\,776,$

$p_7(x) = 720x^6 + 8\,640x^5 + 45\,756x^4 + 137\,512x^3 + 248\,250x^2 + 255\,828x + 117\,649.$

1.3.3 The generating function and the coefficients of the $p_n(x)$ polynomials

The generating function of $p_n(x)$

The *generating function* of a sequence encodes plenty of information about the sequence[2]. We are interested here in the generating function

$$f(x,y)=\sum_{n=1}^{\infty}p_n(x)\frac{y^n}{n!}.$$

of the polynomial $p_n(x)$, introduced in the previous subsection. (Because of the presence of $n!$ in the denominator such a generating function is called *exponential*.)

The recursion

$$p_{n+1}(x)=(1+x)p_n'(x)-(nx+3n-1)p_n(x)$$

results in the following partial differential equation:

$$\frac{\partial}{\partial y}f(x,y)=(1+x)\frac{\partial}{\partial x}f(x,y)-xy\frac{\partial}{\partial y}f(x,y)-3y\frac{\partial}{\partial y}f(x,y)+f(x,y).$$

The general solution of this equation is

$$f_g(x,y)=\frac{g(e^x(x+y+2xy+x^2y))}{1+x}.$$

Here g is an arbitrary function. The particular solution which corresponds to our generating function can be sorted out by checking, say, the coefficient of y in the Taylor series of $f_g(x,y)$. It turns out that this coefficient is

$$e^x(1+x)g'(xe^x).$$

On the other hand, this coefficient must equal $p_1(x)=1$. Therefore, we have that

$$e^x(1+x)c'(xe^x)=1.$$

A quick check should convince the reader that $g'=W'$, whence $g=W$ (up to an additive constant in general, but this constant is now zero). We therefore get that

$$f(x,y)=\frac{W(e^x(x+y+2xy+x^2y))}{1+x}.$$

Notice that

$$f(0,y)=W(y)$$

W. This will be clarified at the end of Section 1.5.

[2]Since the generating function uniquely determines the sequence, we can even say that

follows immediately. Since

$$f(0,y)=\sum_{n=1}^{\infty}p_n(0)\frac{y^n}{n!},$$

we get that the $\frac{p_n(0)}{n!}$ numbers are the Taylor series coefficients of the W function. We further develop this study in Section 1.4. Among others, we will prove that

$$p_n(0)=(-n)^{n-1}\quad(n\geq 1).$$

The coefficients of $p_n(x)$

Apart of the recursion (1.34), there is another way to explicitly represent the coefficients of $p_n(x)$. To this end, we write it in the form

$$p_n(x)=(-1)^{n-1}\sum_{k=0}^{n-1}\beta_{n,k}x^k. \tag{1.36}$$

With the alternating sign we take into account that the p_n polynomials seem to have positive coefficients for odd n, and negative for even n. If we can prove that $\beta_{n,k}$ is always positive, this is justified.

Our statement is the following:

$$\boxed{\beta_{n,k}=\sum_{m=0}^{k}\frac{1}{m!}\binom{2n-1}{k-m}\sum_{q=0}^{m}\binom{m}{q}(-1)^q(q+n)^{m+n-1}.} \tag{1.37}$$

To prove this claim, rewrite (1.33) as

$$p_n(W(x))=(1+W(x))^{2n-1}e^{nW(x)}\frac{\mathrm{d}^nW(x)}{\mathrm{d}x^n}. \tag{1.38}$$

We want to compare the coefficients of x on both sides. Although the right-hand side seems to be complicated, it is just a polynomial of W. First we deal with nth derivative of W. We will learn soon the Taylor series (1.52) of W. From the derivation of both sides of this formula n times, we get that

$$\frac{\mathrm{d}^nW(x)}{\mathrm{d}x^n}=\sum_{m=n}^{\infty}\frac{(-m)^{m-1}}{(m-n)!}x^{m-n}.$$

Substituting this into (1.38), and putting xe^x in place of x, we get

$$p_n(x)=(1+x)^{2n-1}e^{nx}\sum_{m=n}^{\infty}\frac{(-m)^{m-1}}{(m-n)!}x^{m-n}.$$

Next, we make use of the Taylor series

$$e^{nx} = \sum_{m=0}^{\infty} \frac{n^m}{m!} x^m,$$

and multiply this with the rightmost sum. To isolate the coefficients of the powers of x, we still need to expand $(1+x)^{2n-1}$:

$$(1+x)^{2n-1} = \sum_{i=0}^{2n-1} \binom{2n-1}{i} x^i.$$

After following all of these steps, we arrive at (1.37).

More on $\beta_{n,k}$ can be read in the Further Notes at the end of the chapter.

1.3.4 Derivatives of $W(e^x)$

The expression for the derivatives of $W(e^x)$ are somewhat simpler than the ones for $W(x)$ given in (1.33):

$$\boxed{W^{(n)}(e^x) = \frac{q_n(W(e^x))}{(1+W(e^x))^{2n-1}} \quad (n \geq 1).} \tag{1.39}$$

The very same method as above results in a recursion for the q_n polynomials:

$$q_{n+1}(x) = (x^2+x)q_n'(x) - (2n-1)xq_n(x) \quad (n \geq 1), \tag{1.40}$$

with $q_1(x) = x$.

The coefficients of q_n satisfy a nice formula:

$$q_n(x) = \sum_{k=0}^{n-1} \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle (-1)^k x^{k+1}. \tag{1.41}$$

Here $\left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle$ is a so-called *Eulerian number of the second kind*, and satisfies the recursion

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = (k+1)\left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle + (2n-1-k)\left\langle\!\!\left\langle {n-1 \atop k-1} \right\rangle\!\!\right\rangle,$$

with $\left\langle {0 \atop 0} \right\rangle = \left\langle {1 \atop 0} \right\rangle = 1$, and $\left\langle {1 \atop 1} \right\rangle = 0$. Using (1.40) and this recursion one can show the validity of (1.41).

---

it encodes all the information about the sequence. Not all the information is, however, the best accessible through the generating function.

Here are the first five q_n polynomials:

$$\begin{aligned}
q_1(x) &= x,\\
q_2(x) &= x,\\
q_3(x) &= -2x^2 + x,\\
q_4(x) &= 6x^3 - 8x^2 + x,\\
q_5(x) &= -24x^4 + 58x^3 - 22x^2 + x,\\
q_6(x) &= 120x^5 - 444x^4 + 328x^3 - 52x^2 + x,\\
q_7(x) &= -720x^6 + 3\,708x^5 - 4\,400x^4 + 1\,452x^3 - 114x^2 + x
\end{aligned}$$

The Eulerian numbers of the second kind have a nice combinatorial interpretation, see [74] for details.

1.3.5 The differential equation of W

There is a differential equation which can be solved in terms of the Lambert function. This differential equation can be deduced as follows. Let $y(x) = W(x)$, and consider (1.32), which yields the differential equation

$$y'(x)(1 + y(x)) = \frac{y(x)}{x}, \tag{1.42}$$

or the equivalent

$$xy'(x) + xy(x)y'(x) - y(x) = 0.$$

This is a first-order, linear, homogeneous equation. Its solution is

$$y(x) = W(e^c x),$$

where c is an arbitrary constant of integration. The non-homogeneous equation,

$$xy'(x) + xy(x)y'(x) - y(x) = a,$$

has the solution

$$y(x) = -a + (1-a)W\left(\frac{(xe^{a+c})^{\frac{1}{1-a}}}{1-a}\right).$$

The differential equation (1.42) can be generalized such that the Lambert function still suffices to solve the more general equation, too. The solution of the differential equation

$$y'(x)(a + y(x)) = b\frac{y(x)}{x}$$

with arbitrary but non-zero parameters a and b is

$$y(x) = aW\left(\frac{e^{\frac{c}{a}}x^{b/a}}{a}\right).$$

(If a, b or both are zero, the equation has trivial or elementary solutions.)

We shall see situations where the Lambert function comes to our help when solving differential equations coming from practical examples; see Part III of the book.

1.3.6 Some integrals of W

Owing to the fact that $W(x)$ is an inverse function, it is very easy to find its primitive function $\int W(x)$, and a whole host of other integrals as well.

The integral of $W^n(x)$

The method basically goes as follows. If $W(x) = u$, then $x = ue^u$, and $\mathrm{d}x = (1+u)e^u\mathrm{d}u$. Hence

$$\int W(x)\mathrm{d}x = \int u(1+u)e^u\mathrm{d}u.$$

This integral is elementary:

$$\int u(1+u)e^u\mathrm{d}u = e^u(u^2-u+1)+c.$$

We therefore have that

$$\int W(x) = e^{W(x)}(W(x)^2 - W(x)+1)+c.$$

Upon substituting $e^{W(x)} = \frac{x}{W(x)}$:

$$\boxed{\int W(x) = x\left(W(x) - 1 + \frac{1}{W(x)}\right)+c.}$$

Thus, for example,

$$\boxed{\int_0^1 W(x)\mathrm{d}x = \Omega + \frac{1}{\Omega} - 1.}$$

One can deduce a formula for the primitive function of the positive integer powers of $W(x)$. The simplest formulas make use of the *incomplete Gamma function*. This is defined by

$$\Gamma(s,z) = \int_z^\infty t^{s-1}e^{-t}\mathrm{d}t \quad (\Re(s) > 0). \tag{1.43}$$

One has that

$$\int u^n e^u\mathrm{d}u = (-1)^n\Gamma(n+1,-u)+c. \tag{1.44}$$

Using our technique, we write

$$\int W^n(x)\mathrm{d}x = \int u^n(1+u)e^u\mathrm{d}u = \int u^n e^u\mathrm{d}u + \int u^{n+1}e^u\mathrm{d}u.$$

By making use of (1.44), we immediately get that

$$\int W^n(x)\mathrm{d}x = (-1)^n\left[\Gamma(n+1,-W(x)) - \Gamma(n+2,-W(x))\right] + c \quad (n \geq 1).$$

This can further be simplified if we make use of a recurrence of the incomplete Gamma function:

$$\Gamma(s+1,z) = s\Gamma(s,z) + z^s e^{-z}. \tag{1.45}$$

One can then see that

$$\boxed{\int W^n(x)\mathrm{d}x = (-1)^{n+1}n\Gamma(n+1,-W(x)) + xW^n(x) + c \quad (n \geq 1).}$$

Another formula is

$$\boxed{\int W^n(x)\mathrm{d}x = xW^n(x) + x\sum_{k=-1}^{n-1}(-1)^{n-k}\frac{n\cdot n!}{(k+1)!}W^k(x) + c \quad (n \geq 1),} \tag{1.46}$$

which does not involve any other function except W itself.

The integral of $x^nW(x)$

To calculate the integrals of $x^nW(x)$ we shall make use of the integral identities

$$\int u^n e^{nu}\mathrm{d}u = \frac{(-1)^n}{n^{n+1}}\Gamma(n+1,-nu) + c \quad (n \geq 1), \tag{1.47}$$

$$\int u^{n+1}e^{nu}\mathrm{d}u = \frac{(-1)^{n+1}}{n^{n+2}}\Gamma(n+2,-nu) + c \quad (n \geq 1), \tag{1.48}$$

We then have that

$$\int x^nW(x)\mathrm{d}x = \int u^{n+1}e^{(n+1)u}(1+u)\mathrm{d}u =$$

$$\int u^{n+1}e^{(n+1)u}\mathrm{d}u + \int u^{n+2}e^{(n+1)u}\mathrm{d}u =$$

$$\frac{(-1)^{n+1}}{(n+1)^{n+2}}\Gamma(n+2,-(n+1)u) + \frac{(-1)^{n+2}}{(n+1)^{n+3}}\Gamma(n+3,-(n+1)u) + c =$$

$$\frac{(-1)^{n+1}}{(n+1)^{n+2}}\left(\Gamma(n+2,-(n+1)u)-\frac{1}{n+1}\Gamma(n+3,-(n+1)u)\right)+c.$$

Substituting back $u=W(x)$, we have an intermediate formula:

$$\int x^n W(x)\mathrm{d}x=$$
$$\frac{(-1)^{n+1}}{(n+1)^{n+2}}\left(\Gamma(n+2,-(n+1)W(x))-\frac{1}{n+1}\Gamma(n+3,-(n+1)W(x))\right)+c.$$

The recursion (1.45) helps to shorten this formula. An elementary calculation results that

$$\boxed{\begin{aligned}&\int x^n W(x)\mathrm{d}x=\\&\frac{(-1)^n}{(n+1)^{n+3}}\Gamma(n+2,-(n+1)W(x))-\frac{(-1)^n}{n+1}x^{n+1}W(x)+c.\end{aligned}}$$

This formula is not valid when n is a non-integer number, but it can be extended to real values between -1 and zero. This integral will be the Mellin transform of W.

1.3.7 The Mellin transform of W

The *Mellin transform* of a function f is

$$(\mathcal{M}f)(s)=\int_0^\infty t^{s-1}f(t)\mathrm{d}t. \tag{1.49}$$

This transformation has many applications [48], and it is an invertible transformation because the original function f can be determined by its Mellin transform:

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-s}(\mathcal{M}f)(s)\mathrm{d}s.$$

Here $a<c<b$, and a and b are the boundaries of the infinite vertical strip $a<\Re(s)<b$ on which the integral in (1.49) converges.

To determine the Mellin transform of the Lambert function, we shall need the *Gamma function*:

$$\Gamma(s)=\Gamma(s,0)=\int_0^\infty t^{s-1}e^{-t}\mathrm{d}t\quad(\Re(s)>0).$$

With this function, $\mathcal{M}W$ has a surprisingly simple (and nice) expression:

$$\boxed{(\mathcal{M}W)(s)=\frac{(-s)^{-s}}{s}\Gamma(s)\quad(-1<\Re(s)<0).} \tag{1.50}$$

At $s = -1, 0$ the integral as well as the expression on the right-hand side diverges. On other points of the limiting vertical lines of the strip $-1 < \Re(s) < 0$ the integral diverges but the right-hand-side has finite values.

Let us now prove (1.50). We will use the usual correspondences

$$W(t) = u, \quad t = ue^u, \quad \mathrm{d}t = (1+u)e^u \mathrm{d}u.$$

With these

$$(\mathcal{M}W)(s) = \int_0^\infty t^{s-1} W(t) \mathrm{d}t = \int_0^\infty (ue^u)^{s-1} u(1+u)e^u \mathrm{d}u =$$
$$\int_0^\infty u^s e^{us} \mathrm{d}u + \int_0^\infty u^{s+1} e^{us} \mathrm{d}u.$$

These integrals are evaluable in terms of $\Gamma(s)$:

$$\int_0^\infty u^s e^{us} \mathrm{d}u = -(-s)^{-s}\Gamma(s) \quad (-1 < \Re(s) < 0),$$
$$\int_0^\infty u^{s+1} e^{us} \mathrm{d}u = (-s)^{-2-s}\Gamma(s+2) \quad (-2 < \Re(s) < 0).$$

In the last integral, we use the functional equation

$$\Gamma(s+1) = s\Gamma s$$

twice, so that $\Gamma(s+2) = (s+1)s\Gamma(s)$. Then a simple algebraic manipulation gives (1.50). That $-1 < \Re(s) < 0$ must hold comes from the intersection of the conditions of the last two displayed integrals.

1.3.8 The Laplace transform of $W(e^t)$

The Laplace transform is another useful functional transformation with many practical applications. The *Laplace transform* is defined as

$$(\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t.$$

The Laplace transform of the Lambert function would need the evaluation of the integral

$$\int_0^\infty ue^u e^{-sue^u} \mathrm{d}u, \quad \text{and} \quad \int_0^\infty u^2 e^u e^{-sue^u} \mathrm{d}u.$$

But these integrals do not have a known expression in terms of usual special functions. Therefore, we determine the Laplace transform of $W \circ \exp$ instead:

$$[\mathcal{L}(W \circ \exp)](s) = \int_0^\infty e^{-st} W(e^t) \mathrm{d}t =$$
$$s^{s-2}\Gamma(1-s, s\Omega) + \frac{\Omega}{s} \quad (\Re(s) > 0).$$

In this expression Ω is the Omega constant, defined in (1.26).

The proof goes as follows. We substitute

$$W(e^t) = u, \quad \text{so} \quad t = \log u + u, \quad \mathrm{d}t = \left(\frac{1}{u} + 1\right) \mathrm{d}u,$$

and the limit of integration at $t = 0$ becomes Ω ($ue^u = 1$, so $u = \Omega$); and at $t = \infty$ it is $u = \infty$. Hence

$$\int_0^\infty e^{-st} W(e^t) \mathrm{d}t = \int_\Omega^\infty u e^{-s(\log u + u)} \left(\frac{1}{u} + 1\right) \mathrm{d}u =$$

$$\int_\Omega^\infty u^{-s} e^{-su} \mathrm{d}u + \int_\Omega^\infty u^{-s+1} e^{-su} \mathrm{d}u.$$

Both of these integrals can be expressed in term of the incomplete Gamma function (see (1.43)):

$$\int_\Omega^\infty u^{-s} e^{-su} \mathrm{d}u = s^{s-1} \Gamma(1 - s, s\Omega),$$
$$\int_\Omega^\infty u^{-s+1} e^{-su} \mathrm{d}u = s^{s-2} \Gamma(2 - s, s\Omega).$$

To get the wanted formula, we just use the recursion (1.45) to transform $\Gamma(2 - s, s\Omega)$ into $\Gamma(1 - s, s\Omega)$.

1.4 The Taylor series of W_0

The Taylor series of W is determined by the *Lagrange inversion theorem*. Lagrange's formula is often useful when we try to find the Taylor series of an inverse function. It says that if $y = f(x)$ (so that $x = f^{-1}(y)$), then around a point $y_0 = f(x_0)$ the series of $f^{-1}(y)$ is given as

$$f^{-1}(y) = x_0 + \sum_{n=1}^{\infty} \frac{(y - y_0)^n}{n!} \frac{\partial^{n-1}}{\partial x^{n-1}} \left(\frac{x - x_0}{f(x) - y_0}\right)^n \Bigg|_{x=x_0}. \tag{1.51}$$

In our case $f(x) = xe^x = y$, and $x = W(y)$. We choose the point x_0 to be zero (thus $y_0 = 0$ as well). The Lagrange inversion formula for the Lambert W function reads

$$W(y) = \sum_{n=1}^{\infty} \frac{y^n}{n!} \frac{\partial^{n-1}}{\partial x^{n-1}} \left(\frac{x}{xe^x}\right)^n \Bigg|_{x=0}.$$

This is continued as follows:

$$\sum_{n=1}^{\infty} \frac{y^n}{n!} \frac{\partial^{n-1}}{\partial x^{n-1}} \left(\frac{x}{xe^x}\right)^n \Bigg|_{x=0} = \sum_{n=1}^{\infty} \frac{y^n}{n!} \frac{\partial^{n-1}}{\partial x^{n-1}} e^{-nx} \Bigg|_{x=0} =$$

$$\sum_{n=1}^{\infty} \frac{y^n}{n!} (-n)^{n-1} e^{-nx}\big|_{x=0} = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} y^n.$$

We therefore get that the Taylor series of W (substituting the letter x in place of y)

$$\boxed{W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n \quad \left(|x| < \frac{1}{e}\right).} \tag{1.52}$$

A quick application of the ratio test yields that

$$\left| \frac{\frac{(-n-1)^n}{(n+1)!} x^{n+1}}{\frac{(-n)^{n-1}}{n!} x^n} \right| = \left(\frac{n+1}{n}\right)^{n-1} |x| = \frac{\left(1+\frac{1}{n}\right)^n}{1+\frac{1}{n}} |x| \to e|x|.$$

Whence it comes that

$$|x| < \frac{1}{e},$$

as it is written in (1.52).

The W function therefore behaves around $x = 0$ as

$$W(x) = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \frac{125}{24}x^5 - \frac{54}{5}x^6 + O(x^7).$$

It follows that,

$$\boxed{\lim_{x\to 0} \frac{W(x)}{x} = 1.} \tag{1.53}$$

For further reference, we note that the above series represents a part of the *principal branch* of the W function. This principal branch has a much larger domain than the disk $|x| < \frac{1}{e}$; in fact, its domain is the whole complex plane. Hence (1.52) represents only a small portion of W; we address this issue in some detail in the next section and give full details in Chapter 2.

1.5 The number of real solutions of xe^x. The two real branches of W

Let us have a look at the graph of the function $f(z) = ze^z$:

Clearly, $f(z)$ is negative for negative z, zero at $z = 0$, and it is positive for positive z (see Figure 1). A quick analysis of $f'(z) = e^z(1 + z)$ reveals that $f(z)$ strictly increases for $z > -1$, and strictly decreases if $z < -1$. The *global* minimum point is therefore at $z_0 = -1$, where $f(-1) = -\frac{1}{e}$. This also means that f does not take real values less than $-\frac{1}{e}$.

It is therefore seen that

$$ze^z = x \tag{1.54}$$

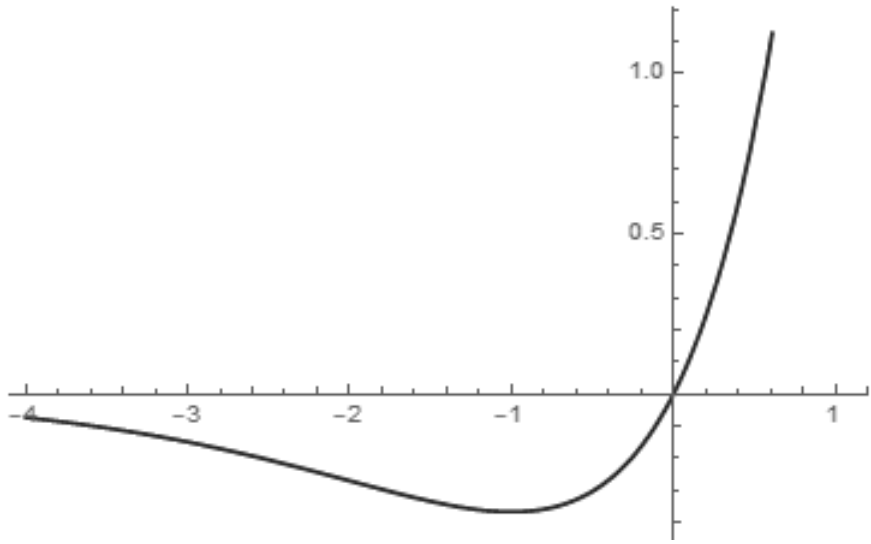


FIGURE 1
The plot of ze^z (z is real). This function has a global minimum at $z = -1$, with value $-1/e$.

has *no* solution when $x < -\frac{1}{e}$; there is *one* solution when $x = -\frac{1}{e}$; it has *two* solutions when $-\frac{1}{e} < x < 0$ (both of these solutions are negative); and it has a *unique* non-negative solution for all $x \geq 0$. Notice that when $-\frac{1}{e} < x < 0$, one of the solutions falls into the interval $]-\infty, -1[$, and the other is in $]-1, 0[$.

By its very definition, the inverse $W = f^{-1}$ serves to give us these solutions. However, W is *not* a function, because at the points of the interval $\left]-\frac{1}{e}, 0\right[$ it takes *two* values.

In such cases, we say that the inverse function has *multiple branches*. These branches serve to distinguish among solutions, and help to *refer to* the different solutions in a consistent way. If properly defined, these branches *become* functions.

A remark is due here. We shall see in Chapter 2 that for all non-zero x, equation (1.54) has *infinitely many complex* solutions. Among these solutions, there is always at most two real solutions. We often refer to the branches that give real solutions as the *real branches*.

Hence, there are two real branches of the Lambert function: for $-\frac{1}{e} \leq x < 0$ the branch $W_{-1}(x)$ will give the smaller negative solution, and $W_0(x)$ gives the other one. Moreover, $W_0(x)$ gives that non-negative solution, too, which is unique for $x \geq 0$. The branch $W_0(x)$ will be called *principal branch* of the Lambert W function. The collection of all the branches *together* constitutes the Lambert W function.

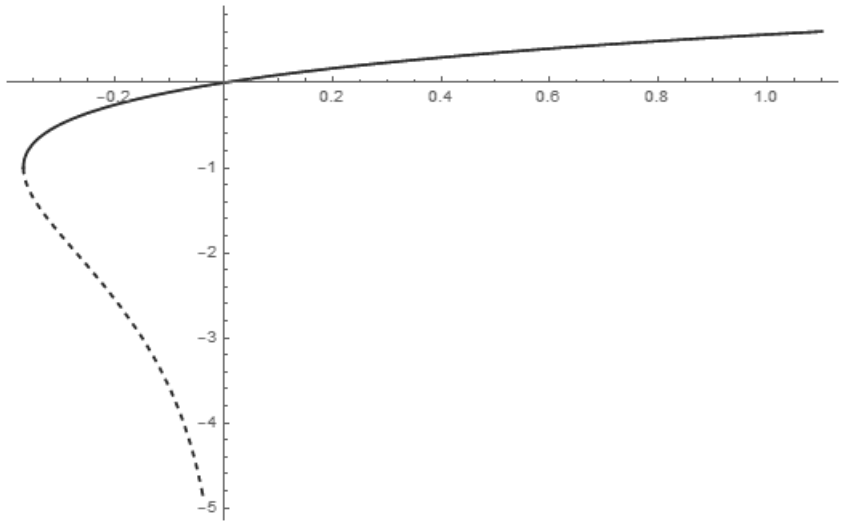


FIGURE 2
The continuous plot is the graph of $W_0(x)$, the dashed plot depicts $W_{-1}(x)$. The two plots connect at the point $(x, y) = (-1/e, -1)$.

For the moment we do not define $W_{-1}(x)$ on the intervals $]-\infty, -\frac{1}{e}[$, and $[0, \infty[$, and we similarly do not define $W(x)$ for x smaller than $-\frac{1}{e}$. These are complex functions, and will be investigated in Chapter 2. Thus

$$W_{-1} : \left[-\frac{1}{e}, 0\right[\to]-\infty, -1], \tag{1.55}$$

$$W_0 : \left[-\frac{1}{e}, \infty\right[\to [-1, \infty[\,. \tag{1.56}$$

The graphs of these two functions (which are the mirror images of the graph of xe^x with respect to the line $y = x$) are given in Figure 2.

The point $x = -\frac{1}{e}$ is a *branch point* of the multi-valued Lambert function: the two branches take the same value in the branch point:

$$W_{-1}\left(-\frac{1}{e}\right) = W_0\left(-\frac{1}{e}\right) = -1.$$

On this graph, this is the point where the continuous and dashed plots meet. (This is completely the same phenomenon as with the square root function: if $f(x) = x^2$, then $f^{-1}(x)$ has two branches: $f_-^{-1}(x) = -\sqrt{x}$, and $f_+^{-1}(x) = \sqrt{x}$; with $f_-^{-1}(0) = f_+^{-1}(0) = 0$.)

It is now clear that the Taylor series (1.52) belongs to the principal branch $W_0(x)$. Indeed, (1.52) is valid on the interval $I =]-\frac{1}{e}, +\frac{1}{e}[$ and it is real there. Only the W_0 branch is real on I, so the validity of our statement is seen.

1.6 Convexity and concavity of W_0 and W_{-1}

Properties of W_0

Now that we know that the real Lambert function is, in fact, two functions, let us study their basic real analytic properties separately.

Figure 2 suggests that the principal branch W_0 is strictly increasing. The sign of the first derivative will justify this. The derivative of W_0 is, by (1.32),

$$W_0'(x) = \frac{W_0(x)}{x(1+W_0(x))}.$$

Note that $\frac{W_0(x)}{x}$ is positive on the whole domain of the real W_0 (at $x=0$ W_0' has only a removable singularity, see (1.53)). Moreover, $1+W_0(x) > 0$, too, on $D_0 = \left]-\frac{1}{e}, \infty\right[$. Therefore

$$\boxed{W_0(x) \text{ is strictly increasing on } \left]-\frac{1}{e}, \infty\right[.}$$

The sign of the second derivative informs us about the convexity of the function. The second derivative of $W_0(x)$ is

$$W_0''(x) = -\frac{W_0(x)^2(W_0(x)+2)}{x^2(W_0(x)+1)^3}.$$

The sign of this expression is determined by the sign of $-\frac{W_0(x)+2}{(W_0(x)+1)^3}$. We have just seen that $1+W_0(x) > 0$ on D_0, and this is even more so for $2+W_0(x)$. The negative sign therefore makes the whole expression negative: $W_0''(x) < 0$ on D_0. Hence

$$\boxed{W_0(x) \text{ is concave on } \left]-\frac{1}{e}, \infty\right[.}$$

This analysis was given through the properties of the inverse function, xe^x, in [22]. In this paper, it was also deduced that not only $W_0(x)$ but also $\frac{x}{W_0(x)}$ is concave, while $\frac{W_0(x)}{x}$ is convex on D_0. The reader can readily check these statements.

Properties of W_{-1}

We check the monotonicity and concavity properties for the other real branch, W_{-1} in the interior of its domain $D_{-1} = \left]-\frac{1}{e}, 0\right[$. The first derivative is

$$W_{-1}'(x) = \frac{W_{-1}(x)}{x(1+W_{-1}(x))}.$$

Clearly, $\frac{W_{-1}(x)}{x}$ is positive on D_{-1}, while $1+W_{-1}(x)$ is negative, so $W'_{-1}(x) < 0$ on D_{-1}. We get that

$$\boxed{W_{-1}(x) \text{ is strictly decreasing on } \left]-\frac{1}{e}, 0\right[.}$$

The second derivative is

$$W''_{-1}(x) = -\frac{W_{-1}(x)^2(W_{-1}(x)+2)}{x^2(W_{-1}(x)+1)^3}.$$

The relevant terms are $(W_{-1}(x)+1)^3$ and $W_{-1}(x)+2$. It is plain that the former is negative on the whole D_{-1}. These negative values turn to be positive with the minus one factor. We get that the sign of $W''_{-1}(x)$ is determined by $W_{-1}(x)+2$.

We have just seen that W_{-1} strictly decreases, so $W_{-1}(x)+2$ changes sign only once. It occurs when

$$W_{-1}(x) = -2, \quad \text{i.e., when} \quad x = -2e^{-2}.$$

On $-\frac{1}{e} < x < -\frac{2}{e^2}$ the values taken by $W_{-1}(x)+2$ are positive, and are negative on $-\frac{2}{e^2} < x < 0$. We therefore get the following fact.

$$\boxed{\begin{array}{l} W_{-1}(x) \text{ is convex on } \left]-\frac{1}{e}, -\frac{2}{e^2}\right[, \\ \text{and it is concave on } \left]-\frac{2}{e^2}, 0\right[. \end{array}}$$

That is, W_{-1} has an inflection point at $x = -2e^{-2}$.

1.7 The Ω constant

The $\Omega = W(1)$ constant was introduced in Subsection 1.2.3. Now that we know more about the branches of the Lambert function, we can more correctly say that Ω is the value at $x = 1$ of *the principal branch* of W. That is,

$$\Omega = W_0(1).$$

Ω satisfies nice equations like

$$\Omega e^{\Omega} = 1, \quad \log\frac{1}{\Omega} = \Omega.$$

Numerical methods – which we will learn about later – help us to find the value of Ω. For 20 digits, its value is

$$\Omega = 0.56714329040978387299996\,86\ldots$$

The last digit is not rounded (the following digit is also six).

1.7.1 Ω is transcendental

The Ω constant is not a rational number; it cannot be written as a fraction $\frac{a}{b}$ with a, b integers. In this respect, it is similar to, for example, $\sqrt{2}$ which is non-rational also. But at least $\sqrt{2}$ can be gotten as the solution of an algebraic equation: the second degree equation

$$x^2 - 2 = 0$$

has two solutions: $x_{1,2} = \pm\sqrt{2}$. Numbers which *are* solutions of some algebraic equation (with rational coefficients[3]) are called *algebraic numbers*. There are numbers which are not algebraic; these are the *transcendental numbers*. Two famous examples of transcendental numbers are e and π.

The Ω constant is also a transcendental number. This statement can be proven by the use of the *Lindemann – Weierstrass theorem* [11]. This theorem (in a form given by A. Baker) says that if $\alpha_1, \ldots, \alpha_n$ are distinct algebraic numbers, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are linearly independent over the algebraic numbers. For our purpose, the statement suffices in the particular case when $n = 1$. Then it states that if α is algebraic, then e^{α} is transcendental. Let $\alpha = \Omega$. If Ω were algebraic, then e^{Ω} would be transcendental. But $e^{\Omega} = \frac{1}{\Omega}$, and the reciprocal of an algebraic number is algebraic, so we get a contradiction. The reason of the contradiction is that our initial assumption (Ω is algebraic) is false.

The Ω constant is transcendental.

This transcendence observation can be strengthened for other values of W, regardless of whether this value is complex or real. A complex number can be algebraic, as the number $i = \sqrt{-1}$ shows. It arises as the solution of $x^2 + 1 = 0$. So, in this respect there is not much difference between real and complex numbers. Another example is: $i\pi$ is transcendental. If it was algebraic, $e^{i\pi}$ was transcendental, but $e^{i\pi} = -1$ (this is the famous *formula of Euler*), an algebraic number. The Lindemann – Weierstrass theorem thus shows that $i\pi$ is transcendental.

After this short detour, we go back to the values of W. Let x be an algebraic number (in particular, it can be an arbitrary rational number $a/b = x$). Then $e^{W(x)} = \frac{x}{W(x)}$. If $W(x)$ were algebraic, $\frac{x}{W(x)}$ were algebraic, too, which would mean that $e^{W(x)}$ is algebraic. But the Lindemann – Weierstrass theorem does not permit the exponentials of algebraic numbers be algebraic. We do not get contradiction only if $W(x)$ is transcendental. We therefore get a nice number theoretical result for a huge class of Lambert function values:

For algebraic values x, the number $W(x)$ is transcendental.

[3] If we dropped the assumption that the coefficients are rational, every real number a could trivially satisfy an equation: $x - a = 0$.

1.7.2 The Adamchik integral (in a generalized form)

V. Adamchik gave a nice integral representation for $\frac{1}{1+\Omega}$:

$$\boxed{\frac{1}{1+\Omega} = \int_{-\infty}^{\infty} \frac{1}{(e^x - x)^2 + \pi^2} \mathrm{d}x.}$$

A more general form is valid [1][4], from which Adamchik's result comes when $a = 1$, $b = 0$:

$$\boxed{\frac{1}{a^2} \frac{1}{1 + W_0\left(\frac{1}{a} e^{-b/a}\right)} = \int_{-\infty}^{\infty} \frac{1}{(e^x - ax - b)^2 + (a\pi)^2} \mathrm{d}x \quad (a > 0, b \in \mathbb{R}).} \tag{1.57}$$

The finding of the integral is a nice application of the *Residue Theorem*, and the knowledge of the branch structure of the Lambert function. Here we use information which will be available only in the second chapter, so the reader might come back to this calculation after being acquainted with that knowledge.

We will calculate the contour integral

$$\int_{\gamma_R} \frac{1}{(e^z - az - b)^2 + (a\pi)^2} \mathrm{d}z,$$

where the contour γ_R is the usual semi-circle contour depicted in Figure 3. Let us determine where the integrand is singular. To this end, we need to solve the equation

$$(e^z - az - b)^2 + (a\pi)^2 = 0,$$

which is equivalent to

$$e^z - az - b = \pm ia\pi. \tag{1.58}$$

This equation is solvable in terms of the Lambert function, see (1.12). What was not mentioned around (1.12) is that there are infinitely many solutions, indexed by an integer k. This integer identifies the branches of W, so (1.12) would better be written as

$$x + b^x = a \quad \Longrightarrow \quad x = a - \frac{W_k(b^a \log b)}{\log b}.$$

This means that the solutions of (1.58) are

$$z_k^{\pm} = \pm i\pi - \frac{b}{a} - W_k\left(\frac{1}{a} e^{-b/a}\right) \quad (k \in \mathbb{Z}).$$

[4]The *Formelsammlung* is a good collection of mathematical facts in German. To find our integral, see the definite integrals section (Liste von bestimmten Integralen), then choose

In order to be able to analyze the positions of these z_k's (we want to see whether they fall inside the curve γ_R), we restrict the argument of W_k to be a positive real number. Thus we want a to be positive, but b can be arbitrary. When the argument of W_k runs through the positive reals, the values of W_k run through a curve on the complex plane, from minus infinity to plus infinity, but the imaginary parts are bounded. From what we will learn in Chapter 2, it turns out that $\Im(W_1(t)) > \pi$ for all $t > 0$, so $\Im(-W_1(t)) < -\pi$, and so

$$\Im(-W_1(t) + i\pi) < 0.$$

This means that z_1^+ is certainly below the real axis, out of the curve γ_R, and this is even more so for z_1^-. Since the ranges of W_k lay above W_1 if $k > 1$, the only singularities which possibly fall inside γ_R are $z_k^\pm$ if $k \leq 0$. When $k < 0$, $\Im(-W_{-1}(t)) > \pi$, and thus even z_{-1}^- is above the real axis, let alone z_{-1}^+ or $z_k^\pm$ for all $k < -1$. We still need to consider $z_0^\pm$. This belongs to the principal branch, and W_0 is real on positive real numbers. This means that z_0^- does not fall into γ_R, but z_0^+ does (if R is not too small).

How many of these singularities fall inside the curve depends, of course, on R. Let this number be $n = n(R)$. After all of these considerations, we have that

$$\int_{\gamma_R} \frac{1}{(e^z - az - b)^2 + (a\pi)^2} \mathrm{d}z =$$

$$2\pi i \sum_{k=-n}^{0} \operatorname{Res}\left(\frac{1}{(e^z - az - b)^2 + (a\pi)^2}, z_k^+ \right) +$$

$$2\pi i \sum_{k=-n}^{-1} \operatorname{Res}\left(\frac{1}{(e^z - az - b)^2 + (a\pi)^2}, z_k^- \right).$$

We calculate the residue with z_k^+:

$$\operatorname{Res}\left(\frac{1}{(e^z - az - b)^2 + (a\pi)^2}, z_k^+ \right) = \lim_{z \to z_k^+} \frac{z - z_k^+}{(e^z - az - b)^2 + (a\pi)^2} =$$

$$\frac{1}{2(e^{z_k^+} - az_k^+ - b)(e^{z_k^+} - a)}.$$

To reach the last step, we used L'Hospital's rule.

Notice that

$$e^{z_k^+} = e^{i\pi - \frac{b}{a} - W_k\left(\frac{1}{a}e^{-b/a}\right)} = -e^{-b/a} e^{-W_k\left(\frac{1}{a}e^{-b/a}\right)} =$$

$$-e^{-b/a} \frac{W_k\left(\frac{1}{a}e^{-b/a}\right)}{\frac{1}{a}e^{-b/a}} = -aW_k\left(\frac{1}{a}e^{-b/a}\right).$$

the section Integrale der Form $R(x, \exp)$. Our method here, however, is a bit different from the one given in [1]. See also [3].

Euler's $e^{i\pi} = -1$ formula was used, together with (1.20). The residue is thus easily seen to be

$$\operatorname{Res}\left(\frac{1}{(e^z - az - b)^2 + (a\pi)^2}, z_k^+\right) = \frac{1}{2\pi i a^2}\frac{1}{W_k\left(\frac{1}{a}e^{-b/a}\right) + 1}.$$

Same calculation shows that at z_k^-

$$\operatorname{Res}\left(\frac{1}{(e^z - az - b)^2 + (a\pi)^2}, z_k^-\right) = -\frac{1}{2\pi i a^2}\frac{1}{W_k\left(\frac{1}{a}e^{-b/a}\right) + 1}.$$

These, therefore, all cancel out except the residue in z_0^+. Hence

$$\int_{\gamma_R}\frac{1}{(e^z - az - b)^2 + (a\pi)^2}\mathrm{d}z = \frac{1}{a^2}\frac{1}{W_0\left(\frac{1}{a}e^{-b/a}\right) + 1}.$$

In the last step we tend with R to infinity, and show that the integral on the semi-circle vanishes, and what only remains is the integral on the real line. Let the semi-circle contour be denoted by C_R. Then, by using $|x|-|y| \le |x+y|$ in the second step,

$$\left|\int_{C_R}\frac{1}{(e^z - az - b)^2 + (a\pi)^2}\mathrm{d}z\right| \le \int_{C_R}\frac{1}{|(e^z - az - b)^2 + (a\pi)^2|}\mathrm{d}z \le$$

$$\int_{C_R}\frac{1}{|e^z - az - b|^2 - (a\pi)^2}\mathrm{d}z \le \int_{C_R}\frac{1}{|e^R - aR - b|^2 - (a\pi)^2}\mathrm{d}z =$$

$$\frac{1}{|e^R - aR - b|^2 - (a\pi)^2}\int_{C_R}1\mathrm{d}z = \frac{R\pi}{|e^R - aR - b|^2 - (a\pi)^2}.$$

The last expression certainly tends to zero when $R \to \infty$, and the proof of (1.57) is finished.

1.7.3 A log-exponential integral

The integral representation that we are going to present is valid not only for $\Omega = W_0(1)$, but also for a range of values of $W_0(x)$:

$$\boxed{W_0(x) = \frac{1}{\pi}\Re\int_0^\pi \log\left(\frac{e^{e^{it}} - xe^{-it}}{e^{e^{it}} - xe^{it}}\right)\mathrm{d}t \quad (-1/e \le x \le e).} \tag{1.59}$$

In particular, for the Ω constant

$$\boxed{\Omega = \frac{1}{\pi}\Re\int_0^\pi \log\left(\frac{e^{e^{it}} - e^{-it}}{e^{e^{it}} - e^{it}}\right)\mathrm{d}t.}$$

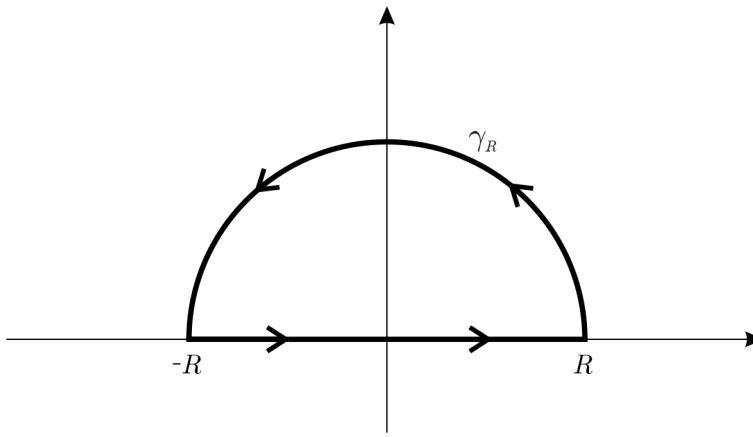


FIGURE 3
The γ_R semi-circle contour used in the text.

This formula was found by the author during working on the manuscript [37]. The proof needs two basic integral evaluations:

$$\int \sin(at+b)\sin(at+d)\mathrm{d}t = \frac{t}{2}\cos(b-d) - \frac{\sin(2at+b+d)}{4a},$$
$$\int \sin(at+b)\sin(ct+d)\mathrm{d}t = \frac{\sin[(a-c)t+b-d]}{2(a-c)} - \frac{\sin[(a+c)t+b+d]}{2(a+c)}.$$

The first integral is the item 2.532/4, and the second one (which is valid only when $a^2 \neq c^2$) is 2.523/1 in [72].

The corollary of these two, when m and n are integers, is the following:

$$\int_0^\pi \sin(mt)\cos(mt)\mathrm{d}t = \begin{cases} \frac{\pi}{2}, & \text{if } m = n; \\ 0, & \text{if } m \neq n. \end{cases}$$

Next, we use the Taylor expansion

$$e^x = \sum_{m=0}^{\infty} \frac{1}{m!} x^m,$$

and De Moivre's formula

$$e^{it} = \cos t + i \sin t \quad (t \in \mathbb{R}).$$

It follows that

$$\Im\left(e^{je^{it}}\right) = \Im\left(\sum_{m=0}^{\infty} \frac{j^m e^{imt}}{m!}\right) = \sum_{m=0}^{\infty} \frac{j^m}{m!} \Im(e^{imt}) = \sum_{m=0}^{\infty} \frac{j^m \sin(mt)}{m!}.$$

With this result we can evaluate the following integral:

$$\Im \int_0^\pi e^{je^{it}} \sin(nt) \mathrm{d}t = \int_0^\pi \sum_{m=0}^{\infty} \frac{j^m \sin(mt)}{m!} \sin(nt) \mathrm{d}t =$$

$$\sum_{m=0}^{\infty} \frac{j^m}{m!} \int_0^\pi \sin(mt) \sin(nt) \mathrm{d}t = \frac{j^n}{n!} \frac{\pi}{2}.$$

This is valid for all real j.

We are now ready to prove (1.59). Substituting $j = -n$ in place of j in our last integral, dividing the two extremal sides by $-n$, and multiplying by $\frac{2}{\pi} x^n$, we get

$$\frac{(-n)^{n-1}}{n!} x^n = -\frac{2}{\pi} \Im \int_0^\pi \frac{x^n e^{-ne^{it}} \sin(nt)}{n} \mathrm{d}t.$$

Summing over $n = 1, 2, \ldots$, and considering the Taylor series (1.52), we get that the left-hand-side is $W_0(x)$, so at this moment we have that

$$W_0(x) = -\frac{2}{\pi} \Im \int_0^\pi \sum_{n=1}^{\infty} \frac{x^n e^{-ne^{it}} \sin(nt)}{n} \mathrm{d}t. \tag{1.60}$$

In the next step we make use of the simply-provable identity

$$\sum_{n=1}^{\infty} \frac{x^n \sin(nt)}{n} = \frac{i}{2} \log \left(\frac{1 - e^{it} x}{1 - e^{-it} x} \right).$$

The sum inside the integral can be put in a single form:

$$\Im \int_0^\pi \sum_{n=1}^{\infty} \frac{x^n e^{-ne^{it}} \sin(nt)}{n} \mathrm{d}t = \Im \int_0^\pi \frac{i}{2} \log \left(\frac{1 - e^{it} x e^{-e^{it}}}{1 - e^{-it} x e^{-e^{it}}} \right) \mathrm{d}t =$$

$$= \frac{1}{2} \Re \int_0^\pi \log \left(\frac{e^{e^{it}} - e^{it} x}{e^{e^{it}} - e^{-it} x} \right) \mathrm{d}t = -\frac{1}{2} \Re \int_0^\pi \log \left(\frac{e^{e^{it}} - e^{-it} x}{e^{e^{it}} - e^{it} x} \right) \mathrm{d}t.$$

Substituting this into (1.60), the proof of (1.59) is established.

That the result is valid only when $x > -1/e$ can easily be seen: the principal branch becomes complex when $x < -1/e$, while on the right-hand side we always have a real number. On the other hand, as $x \to e$, the argument of the logarithm under the integrand tends to plus infinity when $t \to 0$. After the turning point (when $x = e + \varepsilon$) the value of the integral decreases, while $W(x)$ keeps growing monotonically. Thus the representation loses its validity when $x > e$.

1.7.4 A log-trigonometric integral

Another integral for Ω was found by the author (and, independently and earlier, by Kalugin et al. [103]) after encountering with the problem of Nuttall [139] and its solution by Bouwkamp [23]. This integral representation reads as

$$\boxed{\Omega = \frac{1}{\pi}\int_0^\pi \log\left(1+\frac{\sin t}{t}e^{t\cot t}\right)\mathrm{d}t.}$$

In fact, the following representation is valid for the principal branch $W_0(x)$:

$$\boxed{W_0(x) = \frac{1}{\pi}\int_0^\pi \log\left(1+x\frac{\sin t}{t}e^{t\cot t}\right)\mathrm{d}t.} \tag{1.61}$$

From this, the result for Ω comes by setting $x = 1$.

The proof of (1.61) is not hard, once we have the Nuttall – Bouwkamp integral:

$$\int_0^\pi \left[\frac{\sin t}{t}\exp(t\cot t)\right]^\nu dt = \frac{\pi\nu^\nu}{\Gamma(1+\nu)} \qquad \text{for } \nu \geq 0.$$

(Here ν is not necessarily an integer number. Γ is the Euler Gamma function. We only need the fact that $\Gamma(\nu+1) = \nu!$ for non-negative integer ν.) Divide both sides by ν, multiply with $(-1)^{\nu-1}x^\nu$, and sum for all $\nu \geq 1$ integers. On the left-hand-side we get

$$\int_0^\pi \log\left(1+x\frac{\sin t}{t}e^{t\cot t}\right)\mathrm{d}t.$$

On the right-hand side, we have the following sum:

$$\pi\sum_{\nu=1}^{\infty}\frac{(-\nu)^{\nu-1}}{\nu!}.$$

The sum is nothing else but the (1.52) Taylor series of the principal branch of the Lambert W function. Our result is therefore proven.

Although we used the Taylor series to connect the integral to W_0, numerical calculations show that (1.61) is valid on a large subset of $\mathbb{C}$ (also in $x = 1$ although this point does not belong to the convergence domain of the Taylor series). The exact set on which (1.61) is valid is not known.

Notice that the substitution $x = -1/e$ and (1.5) yields a nice integral for π:

$$\boxed{\pi = \int_0^\pi \log\left(\frac{ex}{ex - e^{x\cot x}\sin x}\right)\mathrm{d}x.}$$

Further notes

1. **Some more words on history.** The equation $xe^x = a$ or its variants like $x^x = a$ were studied by several people during the last centuries before the W function was formally introduced. See the pages of Ramanujan's "Lost Notebook" around Example 3(iii) [18]. See also Wright's papers mentioned in the Prelude, or Eisenstein's article [55]. Knoebel [107] studied the equation $x^y = y^x$ and compiled a very good and extensive list of bibliography.

2. **Additional differential equations.** We saw in Subsection 1.3.5 that several differential equations can be solved in terms of W. There are many more such differential equations. A good collection of these can be found in [52].

3. **Properties of $\beta_{n,k}$.** The $\beta_{n,k}$ coefficients of the p_n polynomials (see (1.36)) have nice properties that were discovered in [102]. The authors presented (1.37), and other connections between $\beta_{n,k}$ and counting sequences. Among others, it is proved there that for $n \geq 3$, both of the sequences $(k!\beta_{n,k})_{k=0}^{n-1}$ and $(\beta_{n,k})_{k=0}^{n-1}$ are log-concave. A sequence a_k is log-concave, if $a_{k-1}a_{k+1} \leq a_k^2$. A consequence of this fact (for non-negative sequences) is that a log-concave sequence strictly grows, reaches a maximum in at most two positions, then it strictly decreases. (This fact can directly be observed on p. 16, after omitting the negative signs.)

4. **W_0 is a Bernstein function on $]0, \infty[$.** Another property of the $(\beta_{n,k})$ sequence is proved to be important. It can be shown that all the $\beta_{n,k}$ numbers are positive, see Theorem 3.4 in [102]. This observation has an interesting consequence. It yields that the principal branch W_0 is a *Bernstein function*. A function $f :]0, \infty[\to]0, \infty[$ is a Bernstein function if it is infinitely many times differentiable, and the derivatives alternate in sign:

$$(-1)^{n-1} f^{(n)}(x) \geq 0 \quad (x > 0, n \geq 1).$$

   Putting together (1.33), (1.36), and the positivity of $\beta_{n,k}$ we get that W_0 is a Bernstein function. Bernstein functions enjoy nice properties and have many applications [157].

5. **The Lambert function is not Liouvillian.** The question arises whether W is not a genuinely new function but can be expressed in terms of "more elementary" functions. It turns out [26] that W is not a composition of a finite number of exponentials, constants, roots, and integrals of these but rather a genuinely new function.

6. **Problem.** Determine the domain of validity of (1.61).

2

The branch structure of the Lambert W function

In the previous chapter, we mentioned several times that the Lambert function can take complex values. The equation $xe^x = a$ is not solvable in the set of real numbers if $a < -1/e$ (since xe^x has a global minimum at $x = -1$ with value $-1/e$), but it *is* solvable if we enable complex numbers to appear. If we do not restrict ourselves to the reals, a whole new world opens up. This chapter is dedicated to the thorough study of the complex Lambert function.

Before we start this study, however, it is instrumental to see everything on a simpler and more familiar function. We therefore first discover the behavior of the exp-log pair before going to see the $x \exp$-W pair.

2.1 The exponential function

2.1.1 The mapping properties of exp

To know how the inverse of the exponential behaves, we need to see the mapping properties of exp. The exponential function is an *entire function*; its Taylor series

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges on the whole set $\mathbb{C}$ to a finite value. It is also well known that

$$\exp(-z) = \frac{1}{\exp(z)}.$$

From these two facts, an important observation follows: $\exp(z)$ can nowhere be zero. Indeed, if exp were zero in a point $z_0 \in \mathbb{C}$, then we would have that $\exp(-z_0) = \frac{1}{\exp(z_0)}$ is infinite, but we have just seen that exp takes finite values in all the points of the complex plane. Thus,

$$\exp : \mathbb{C} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}.$$

Picard's theorem says that a non-constant entire function can miss *at most* one value (see, for example, [110]). The exponential function, therefore, takes

DOI: 10.1201/9781003168102-2

every complex number except zero. We are going to show that it takes every non-zero complex number *infinitely many often*.

Recall *De Moivre's formula*:

$$\exp(it) = \cos(t) + i\sin(t) \quad (t \in \mathbb{R}).$$

Therefore, $|\exp(it)| = \sqrt{\cos^2(t) + \sin^2(t)} = 1$, and the image of the imaginary axis by exp is the unit circle, spanned infinitely many times. Now analyze the more general expression with $z = s + it$:

$$\exp(z) = \exp(s + it) = \exp(s)\exp(it) = \exp(s)[\cos(t) + i\sin(t)]. \tag{2.1}$$

The vertical axis with abscissa s is mapped onto the circle with radius $\exp(s)$ (and with center in the origin[1]), countably infinitely many times. This already shows our claim: every non-zero complex point is taken by exp infinitely many times.

Another proof

Recall the *Great Picard Theorem*: if an analytic function has an essential singularity at a point z, then in any neighborhood around z, the function takes every value with at most one exception infinitely often.

It is an immediate corollary of this theorem that a non-polynomial entire function takes every value with at most one exception infinitely many times. Indeed, if $f(z)$ is a non-polynomial analytic function, then $f(1/z)$ has an essential singularity at 0. From the Great Picard Theorem, it now follows that $f(1/z)$ (and thus f itself) has the said property.

See [35] for the proof of the Great Picard Theorem.

2.2 The definition of the logarithm

After the acquaintance with the exponential, we can introduce its inverse. We have just seen that every point on the complex plane has *infinitely many pre-images*:

$$|\exp^{-1}(w)| = |\{z \in \mathbb{C} : \exp(z) = w\}| = \aleph_0 \quad (w \in \mathbb{C}^*).$$

If we want to know which z points are mapped to $w(\neq 0)$ by exp, we recall (2.1) and calculate the radius of the circle which goes through w (this is simply $|w|$), and the angle of w. The logarithm of the radius, i.e., $\log|w|$ will

[1] In this section, if no otherwise said, the circles are always centered at the origin.

be $\Re(z)$. Here there is no problem, the positive real number $|w|$ has a unique real logarithm.

The angle of w will be the imaginary part of z, but here we must be careful. Since

$$\exp(it) = \cos(t) + i\sin(t) = \cos(t + 2k\pi) + i\sin(t + 2k\pi) = \exp(i(t + 2k\pi)),$$

so the imaginary part of the pre-image z is determined only up to modulo 2π.

To proceed, we recall the *argument function* arg which gives the angle of w such that it falls in the interval $]-\pi, \pi]$. One could think that it would be of more sense to have the values on $[0, 2\pi[$; but in this case, when we go around a circle of some positive radius, we will have a jump in the values of the arg function at the positive axis when we reach it from below (2π to zero) or from above (0 to 2π). Since this jump is inevitable if we want the arg function to take a unique value, it is preferred to have this jump on the negative real axis. This is also dictated by the long tradition.

It is thus seen that all those z's will constitute the pre-image of w for which $\Re(z) = \log|w|$, and $\Im(z) = \arg(w) + 2k\pi$ $(k \in \mathbb{Z})$. Hence, if $w \neq 0$, the pre-image of w is a set:

$$\exp^{-1}(w) = \{z \in \mathbb{C} : z = \log|w| + i(\arg(w) + 2k\pi),\ k \in \mathbb{Z}\}. \tag{2.2}$$

The value $\exp^{-1}(0)$ is not defined[2]. The inverse of the exponential function, the *logarithm*, as we now see, is not a function but an infinite set of functions: for all integer k there is a well defined inverse of exp. These are the *branches* of the logarithm:

$$\log_k(w) = \log|w| + i(\arg(w) + 2k\pi) \quad (w \in \mathbb{C}^*,\ k \in \mathbb{Z}). \tag{2.3}$$

Their ranges are horizontal strips of height 2π (see Figure 4):

$$\log_k : \mathbb{C}^* \to \{z \in \mathbb{C} : \Re(z) \in \mathbb{R},\ \Im(z) \in]-\pi + 2k\pi, \pi + 2k\pi]\}.$$

The sub-index k is not the base of the logarithm, but a *branch index*. (We will not use logarithm of base different from e, so there will be no confusion.) The logarithm with which we met in high school is the branch with $k = 0$, and it is the *principal branch*, denoted usually by log, without index:

$$\log(w) = \log|w| + i\arg(w) \quad (w \in \mathbb{C}^*).$$

This is the only branch which takes real values. Usually, the branch which has the (preferably positive) reals in its domain, or at least "the most of it" is the principal branch. Compare this with with the principal branch W_0 of the logarithm function.

[2]The point $w = 0$ does not belong to the range of exp. We naively could think that by defining $\exp^{-1}(0)$ in a suitable way, we could remedy this problem. This should be done such that log becomes continuous. This is impossible, as it turns out to be obvious after scrutinizing the logarithm more profoundly, see Subsection 2.3.2.

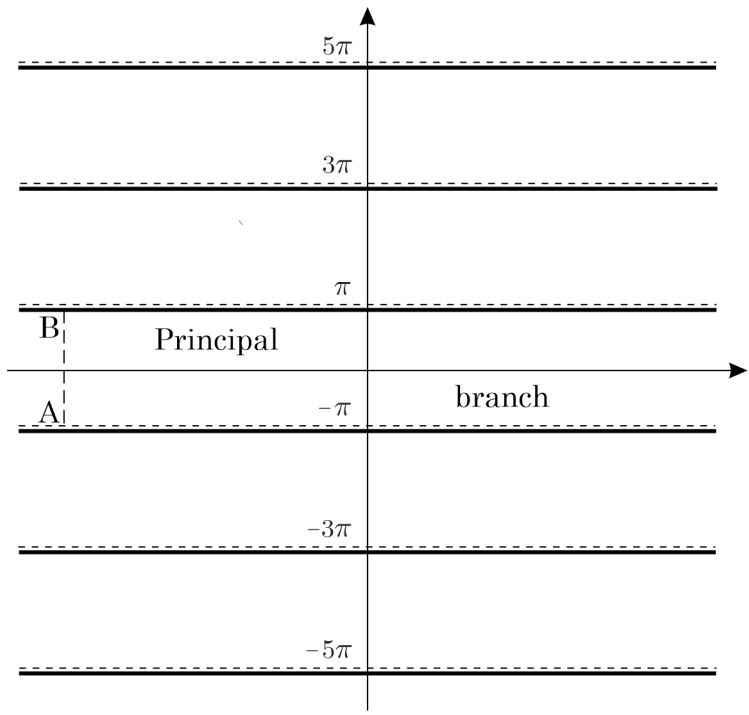


FIGURE 4
The range of the branches of the logarithm. The thick horizontal lines limit the ranges, and the dashed lines show the *branch closure*. For example, the range of the principal branch ($-\pi < \Re(z) \leq \pi$) does not contain the horizontal line at $-\pi$, so we draw a dashed line inside the range, right above the horizontal line at $-\pi$. In turn, the horizontal line at π does belong to the principal branch. These come from the definition of the arg function. The points A and B will be explained in Subsection 2.3.2.

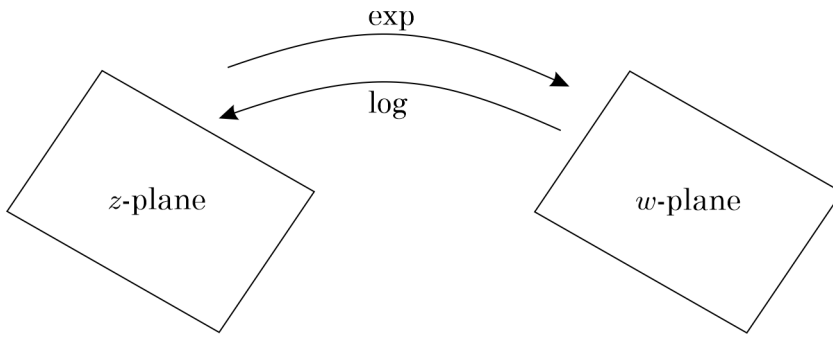


FIGURE 5
The exponential function maps the z-plane into the w-plane, the logarithm maps the latter into the former.

2.3 Branch cuts, covering maps, and Riemann surfaces

To prepare for the analysis of the branch structure of the Lambert function, we introduce some notions which facilitate talking about mappings and their inverses.

2.3.1 The z-plane and the w-plane

When considering equations like

$$\exp(z) = w \quad \text{and} \quad z = \log(w),$$

it is beneficial to distinguish between the z-plane and the w-plane. This makes the references easier and avoids confusion. In this case, the z-plane is the domain of exp; this plane is mapped to the w-plane by the same function. Set theoretically,

$$\text{the } z\text{-plane is } \mathbb{C}, \text{ and the } w\text{-plane is } \mathbb{C}^*.$$

In turn, log, the inverse of exp, maps the w-plane back to the z-plane. This is shown schematically in Figure 5.

Thus, Figure 4 shows the z-plane, partitioned by the ranges of log. All of the individual horizontal strips are mapped by exp to the whole $\mathbb{C}^*$, and $\mathbb{C}^*$ is mapped onto the horizontal strips by the corresponding branches of log.

2.3.2 Branch cut and branch point

It is useful to get a deeper insight into what happens with the lines limiting the horizontal strips of Figure 4 in the z-plane when mapped by exp. We saw

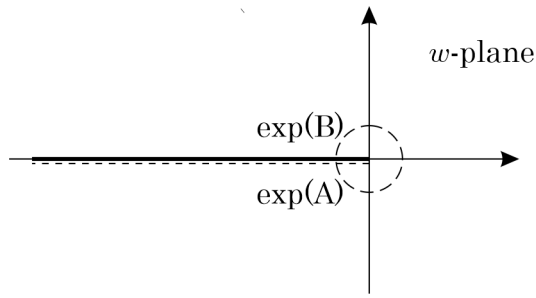


FIGURE 6
Illustration of the branch cut. The meaning of the points exp(A) and exp(B) are detailed in the text.

earlier that horizontal lines are mapped into half-lines from the origin in the w-plane with angles determined by the ordinate of the line. The ordinates limiting the branches are the odd integer multiples of π by an odd integer, so these lines are all mapped onto the negative real axis. The negative axis in the w-plane has therefore a distinguished role. We have a name for it: the negative real axis is the *branch cut* of the logarithm; it is the set in the w-plane corresponding to the branch limiting lines in the z-plane.

If a horizontal line L in the z-plane is parametrized by $s + i(\pi + 2k\pi)$ ($k \in \mathbb{Z}$ fixed, $s \in \mathbb{R}$), then

$$\exp(L) = \{-e^s : -\infty < s < \infty\}.$$

The limiting point, when $s \to -\infty$, is the origin, the other one (when $s \to \infty$) does not belong to the finite plane. The origin is also the limiting point of the branch cuts. Because of this special role, the origin is the *branch point* of the logarithm.

Let us consider a circle going around the origin in the w-plane, as shown in Figure 6. This circle is the image of the vertical line segment A-B in the z-plane (and also the image of any line segments arising by shifting A-B up or down by 2π), as seen in Figure 4. The interiors of the segment A-B and the point B belong to the principal branch, but the point A does not. When traversing on the segment A-B on the z-plane, the exponential runs on the circle counter-clockwise in the w-plane. The images of the points A and B fall onto the branch cut. By the closure of the principal branch of the logarithm, the circle is continuous at its endpoint, exp(B). The branch cuts are usually chosen in such a way that this *counter-clockwise continuity* (CCC) holds.

Regardless of the radius of the circle, it always meets three branches (the principal branch, and $\log_{\pm 1}$). More generally, regardless of what kind of curve we use to connect two arbitrary points on the horizontal lines bounding the principal branch (or any other branch) in the z-plane, the image of

this curve by the exponential always encircles the origin in the w-plane in counter-clockwise direction, see Figure 7.

2.4 Riemann surfaces

2.4.1 Motivation

It is an inconvenience that the logarithm and the Lambert function (and, in general, the multi-valued functions, such as $\sqrt{z}$, arcsin, etc.) need a branch index to specify which concrete value we are talking about. Another problem is that complications might arise when we try to use "identities", like

$$\log(xy) = \log x + \log y.$$

This equation indeed holds for positive real x and y, but not in general. Take, for example, $x = y = -i$. Then (do not forget that log without index denotes the principal branch)

$$\log(xy) = \log((-i)^2) = \log(-1) = \log|-1| + i\arg(-1) = i\pi.$$

On the other hand,

$$\log(x) + \log(y) = 2\log(-i) = 2(\log|-i| + i\arg(-i)) = 2i\left(-\frac{\pi}{2}\right) = -i\pi.$$

These problems motivated Bernhard Riemann to lay down a theory for the better understanding of multi-valued functions. His efforts culminated in a 1851 doctoral thesis, *Foundations for a General Theory of Functions of a Complex Variable*. Hermann Weyl further developed Riemann's ideas, and defined the abstract Riemann surfaces in his 1913 book *The Concept of a Riemann Surface*. Weyl wrote [183]:

> I shared his [Felix Klein's] conviction that Riemann surfaces are not merely a device for visualizing the many-valuedness of analytic functions, but rather an indispensable essential component of the theory; not a supplement, more or less artificially distilled from the functions, but their native land, the only soil in which the functions grow and thrive.

All of these serve as a very good motivation for us to take a look at the general definition of Riemann surfaces, and apply what we have learned to the logarithm first, and then to the Lambert function. We will see that both of these functions can be represented as a single-valued function, by some "gluing process". Riemann's idea was the following. Instead of taking much care about the range of multi-valued functions ($\log_{-1}$ maps $\mathbb{C}^*$ to the

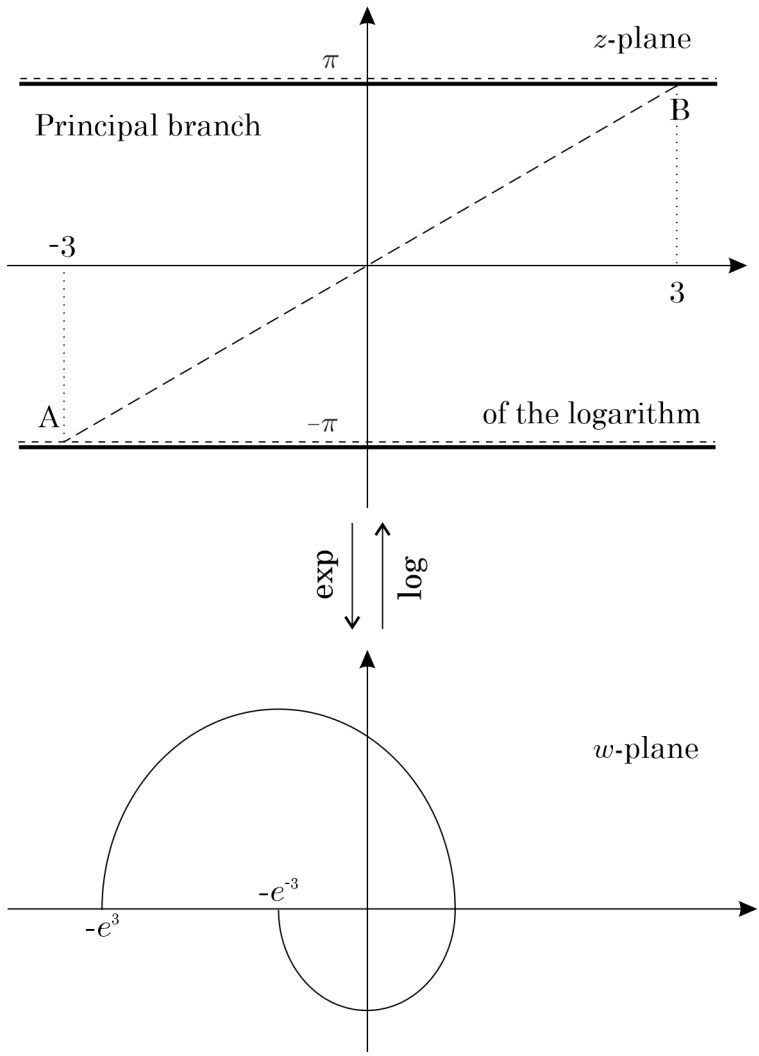


FIGURE 7
The slanted line in the z-plane is mapped by the exponential to a spiral. The initial point of the spiral is $\exp(A) = -e^{-3}$ and its enpoint is $\exp(B) = -e^{3}$. Passing through the slanted line upward, the distance of the image point by exp grows, while its argument goes from $-\pi$ to π.

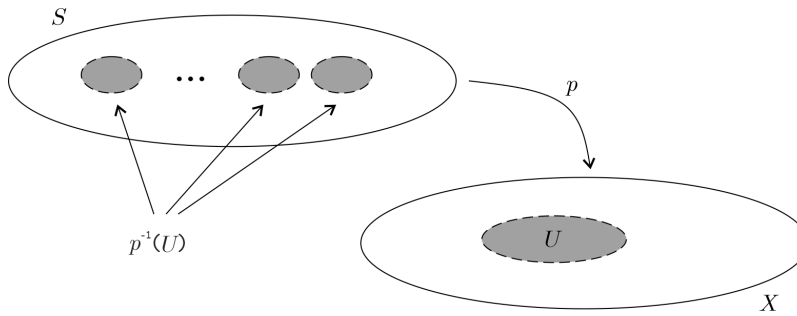


FIGURE 8
Illustration of a covering: p is the covering map, X is the base space, and S is the total space. The pre-image of U by p is a collection of disjoint subsets in S, which are called "sheets".

horizontal strip with imaginary part in $]-3\pi, -\pi]$, etc.), one could *extend the domain* of the function. If, in place of $\mathbb{C}^*$, we take an infinite copy of this set, the new single-valued logarithm will have enough space to map from this infinite copy to the whole $\mathbb{C}$, not only to the horizontal strips. Of course, if we want this new function to be continuous, we need some connection between the copies of $\mathbb{C}^*$, i.e., we need to "glue" these copies together in a meaningful way. The natural framework for all of this is the theory of Riemann surfaces and coverings.

2.4.2 Covering maps

We start with the formal definition of covering maps and the related notions, then we apply these to the exponential function and its inverse.

Let X and S be two topological spaces. The *covering map* is a continuous and surjective function $p : S \to X$ with the following property. For every $x \in X$ there is an open (connected) neighbor $U \subset X$ containing x, such that $p^{-1}(U)$ is a union of disjoint, open sets, the *sheets* in S, and these sheets are mapped homeomorphically onto U by p. X is the *base space*, and S is the *total space* of the covering in question. The pre-image $p^{-1}(\{x\})$ of $x \in X$ by the inverse of p is a discrete subset of the total space S. Such sets are called *fibers*. The situation (without fibers) is depicted in Figure 8.

2.4.3 The notion of a surface

The topological space S will be called *surface*, if it is a *Hausdorff space*, and every point $s \in S$ has an open neighborhood homeomorphic to the open unit disk of $\mathbb{C}$. This latter requirement says that the surface S locally looks like a part of the complex plane. (That we denoted the total space and a generic

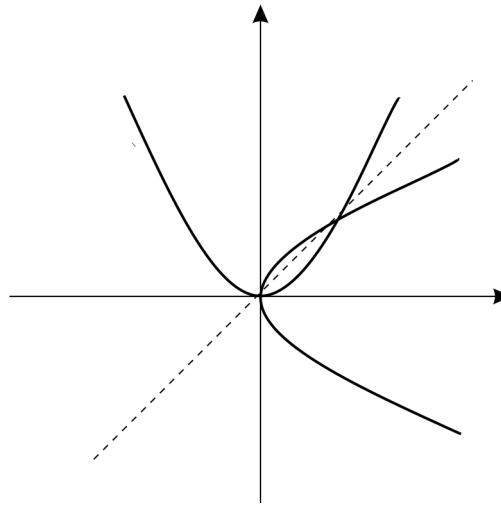


FIGURE 9
The graphical connection between the function $f(x) = x^2$, and its inverse. The inverse of the upward function is drawn by mirroring the points of the original function through the dashed line $y = x$.

surface with the same letter is not a coincidence. In our setting, these will indeed coincide.)

The definition of a Riemann surface is necessarily more complicated, because it introduces coordinate charts on (a family of open sets of) the surface, and requires that these charts satisfy some natural properties. We will not need these details later, so we do not go into details here (but we will still call our constructed surfaces *Riemann* surfaces, because this is the common reference in the literature, see, for example [42]). Instead, we show two simple examples of a Riemann surface. Certainly the simplest is the complex plane itself. A second example is the torus, where the unit square's two opposite sides are glued together such that first we have a "tube" of length one, then we glue the two ends of the tube. The torus clearly satisfies the requirements of a surface.

Next, we see the more complicated example of the Riemann surface of the logarithm. For the sake of better understanding, we first show a heuristic argument.

2.4.4 The Riemann surface of the logarithm – heuristic way

In high school we learned that the inverse of a function can be depicted by mirroring the graph of the function through the 45-degree line $y = x$. (See Figure 9).

If we use this technique, from the graph of xe^x (Figure 1) we get the graph of W_{-1} and W_0 (Figure 2).

We want to extend somehow this technique to the complex plane. If we scrutinize this trick of mirroring, we realize that what we do is the interchange of the axes: $y = f(x)$ means that we map the value of f at x above or below x, and, after mirroring we put the values of f on the x-axis, while the value(s) of the inverse will be drawn over and above the given point. We can do this with complex numbers also, but the problem is that we need four dimensions to do this. Indeed, let $w = \exp(z)$ with $z = s + it$, and $w = u + iv$. We have four real parameters, s, i, u, v. With the aid of a computer, we can render three-dimensional images, and the value in the fourth dimension will be encoded in a *gray value*.

That is, on our graph, the horizontal (w-)plane is

$$\xi = \Re(w) = \Re(\exp(z)) = \exp(s)\cos(t), \quad \text{and}$$
$$\eta = \Im(w) = \Im(\exp(z)) = \exp(s)\sin(t).$$

We introduced ξ and η to make the computer coding easier. Over this plane we want to place the values of the logarithm, the $z = s + it$ elements. As we cannot directly visualize two dimensions over a plane, we have two choices: (1) the real values (variable s) of log will go on the vertical axis, and we encode the imaginary values (t-values) with shades of gray, or (2) the imaginary values will go on the vertical axis, and we encode the real values with shades of gray.

After a moment of thinking, we realize that the first option would not visually work: all the branches have the same real part, (see (2.3)), while every branch has its own imaginary part, different from all the others. This would mean that over every point in the w-plane ((ξ, η)-computer plane) we would have a single point (the unique real part), but this point would have *infinitely many different colors* (a separate gray value for each of the imaginary parts). We therefore choose option 2.

On the language of computers, we take therefore the three-dimensional (ξ, η, ζ)-space, ξ and η will be given as above, we set ζ to be t, and then color this point with a gray value based on s. We choose the gray value such that the smaller the value of s, the darker the gray value.

With the computer package *Mathematica* we can do the above by typing in the command

$$c{:=}5$$
$$\text{ParametricPlot3D}\left[\{\exp(s)\cos(t), \exp(s)\sin(t), t\}, \{t, -4\pi, 4\pi\}, \{s, -c, c\},\right.$$
$$\text{ColorFunction} \to \text{Function}\left[\{\xi, \eta, \zeta, t, s\}, \text{GrayLevel}\left[\frac{\text{Abs}[s]}{c}\right]\right],$$
$$\left.\text{ColorFunctionScaling} \to \text{False}\right]$$

The ParametricPlot3D command plots a three dimensional surface in the (ξ, η, ζ)-space (*Mathematica* uses the x, y, and z symbols, but z denotes something else in our context). The points are determined by the first three values in curly brackets, then come the two parameters for the surface (here t and s). The next parameter gives how the surface is colored. The color is now determined by the absolute value of s. By setting the ColorFunctionScaling option to false, we tell *Mathematica* that we do not want the colors rescaled (to lie between 0 and 1); instead, we use the colors exactly as specified.

The execution of the command results in the plot that is given in Figure 10.

The spiral shape is due to the arg function. Going up/down in the ζ computer-coordinate, the argument function linearly increases/decreases. Revolving 360 degree (2π radius) around the origin, we make a period on the spiral.

The arrow points to the horizontal line which belongs to the principal branch over the line $\eta = 0$ (as it is seen in the bottom). When the computer coordinate η is zero, we have $\sin(t) = 0$, and $t = 0$ in the principal branch. This horizontal line therefore encodes the "high school"-logarithm function. Going outward from the center of the plot, the points of the line are colored with gradually brighter gray values, signaling that the real part of the logarithm increases there. We see now that the high-school log function is just a tiny part of a big picture!

This surface revolves around zero in the t-coordinate (computer ζ) infinitely many times, it is continuous outside of the origin, and establishes a one-to-one connection between the punctured complex plane and the different branches of the logarithm function. This surface is the *Riemann surface of the logarithm*.

2.4.5 The Riemann surface of the logarithm – exact way

As we said above, we want to extend the domain of the logarithm so that the domain has "sufficiently many" elements to map continuously and bijectively to the whole complex plane, not only to the horizontal strips. It is natural to think that a covering map will help here, since the inverse of a covering map "multiplies" the base set. We therefore attempt to construct this domain extension of $\mathbb{C}^*$ with the aid of coverings.

Let the base space be $X = \mathbb{C}^*$, and let the total space S be a surface. Since S intents to be the extended domain of log, we want S to be mapped by a function L to the whole set $\mathbb{C}$ (the union of the ranges of the branches of log) such that the image of each sheet of S by L corresponds to the image of a particular branch of the logarithm (a horizontal strip in $\mathbb{C}$). This aim can be reached via the covering map $p : S \to \mathbb{C}^*$. Our aim is equivalent with requiring the following. Let S be mapped by L onto $\mathbb{C}$ and the images of L be further mapped by exp to elements in $\mathbb{C}^*$ exactly as the covering p maps

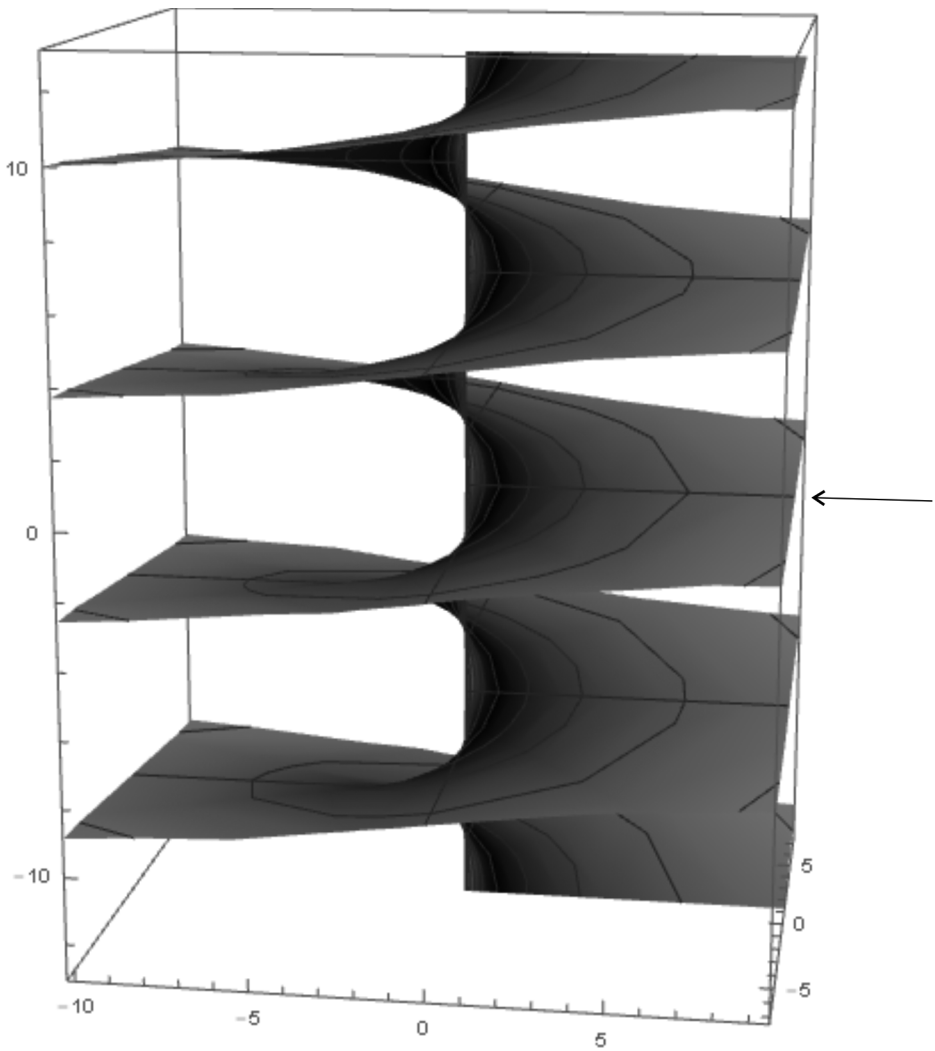


FIGURE 10
The Riemann surface of the logarithm. The arrow points to a line which belongs to the classical, real logarithm function whose values are pictured with gray values. Going toward the origin, the colors on the line are darker, meaning that log takes large negative values there. The lighter the color, the geater the value of the logarithm. In particular, the gray "loarithmically fades out" going outward of the origin, and it becomes white at infinite.

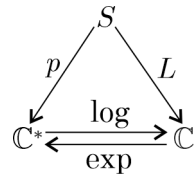


FIGURE 11
S is the Riemann surface of the logarithm, L is the continuous, single-valued logarithm, p is the covering which maps the Riemann surface S to $\mathbb{C}^*$. We require that $\exp \circ L = p$, as it is detailed in the text.

the points of S *directly* onto $\mathbb{C}^*$. In short,

$$\exp \circ L = p. \tag{2.4}$$

The relationship among S, $\mathbb{C}^*$ and $\mathbb{C}$ and their maps is shown in Figure 11. The requirement (2.4) can be expressed in algebraic terms: (2.4) is equivalent to the commutativity of the diagram of Figure 11, we require this diagram (without the log arrow!) to be *commutative*.

How should the surface S be imagined? The inverse of the covering p maps every point in $\mathbb{C}^*$ back to S with infinite multiplicity. This shows that S should be considered as an infinite copy of the topological space $\mathbb{C}^*$. Indeed, if at the end of the day we want everything single-valued, these copies must be distinguishable. On the other hand, we want S not only to be a topological space as the notion of covering minimally requires, but we want it to be a surface, so S must locally be homeomorphic to $\mathbb{C}$. These altogether show that S must be taken as an infinite copy of $\mathbb{C}^*$.

We can make the situation more concrete by appealing to the requirement (2.4). Since p is continuous, L continuously maps the constituting copies of S onto the strips. As these strips join via horizontal lines, the copies of $\mathbb{C}^*$ in S must also be joined. The horizontal lines correspond to the negative axis (branch cut) in $\mathbb{C}^*$, so we glue the copies of $\mathbb{C}^*$ via these branch cuts (hence the name actually).

The cutting and gluing is as follows. We cut up the plane $\mathbb{C}^*$ as it is shown on Figure 12. Let this plane be P_0. Next, we took another copy P_1 cut the same way, place it above P_0, and glue P_1's dashed half-line to the solid line of the sheet P_0. Now take another copy, P_{-1}, place it below P_0, and glue the open half-line of P_0 to the solid line of the sheet P_{-1}. Continue this process upward and downward *ad infinitum*. This will be the Riemann surface S of the logarithm. As it is seen, it is the same as the plane we constructed heuristically in the previous subsection.

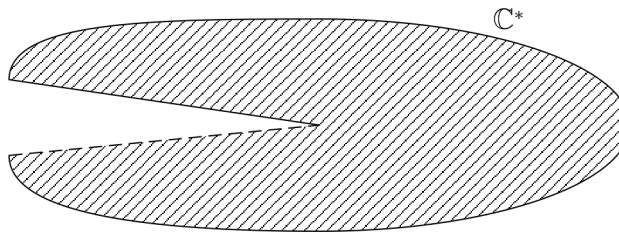


FIGURE 12
S is constituted by an infinite copy of the punctured plane $\mathbb{C}^*$. Each plane is cut as it is shown here. The dashed line does not belong to the set.

2.4.6 The monodromy group of the Riemann surface of the logarithm

Figures 4 and 6 and the text around them explained that a circle around the origin in the w-plane is mapped by a given branch of log such that we go through the whole strip belonging to this branch (see the A-B line segment in Figure 4). If we traverse the circle once again in the w-plane, we get the very same situation.

Now that we are familiar with the Riemann surface of the logarithm, let us see how the situation changes when we want to draw a circle around the origin. A short gaze at Figure 12 shows that the start and endpoint of the circle moved apart from each other. It is not a circle anymore but a *curve*: a *spiral*. If we wish, we can continue on this spiral upward or downward, stepping into another sheet of $\mathbb{C}^*$.

Notice that even to start to draw the curve, we have to *specify* which sheet we choose, because there are infinitely many of them. We had no such option on the single space $\mathbb{C}^*$.

As this is happening in the domain of the one-valued logarithm L, the logarithm of such a curve is passing through *neighboring* branches of the classical logarithm.

There are multi-valued functions for which the branches are taken in complicated orders when going around the branch points on a continuous curve. One can attach a group structure to the surface and to a fixed branch point which captures these branch-orders via permutations. This group of the surface is its *monodromy group*.

The Riemann surface of the logarithm has only one branch point, so there will be one single monodromy we want to determine. As we have already observed, going on our curve around the branch point zero, we touch the branches (which are indexed by the integers) *in succession*. Therefore, the monodromy group of the Riemann surface of the logarithm is the *infinite cyclic group* $\mathbb{Z}$.

2.5 The branches of the Lambert function

After seeing the simpler example of the logarithm, we are now ready to consider the branch structure of the Lambert W function.

2.5.1 The partition of the z-plane

The function $z\exp(z)$ has a more complicated behavior than $\exp(z)$ has. To see this, let $z = s + it$, and $w = z\exp(z)$. Then

$$\Re(w) = e^s(s\cos t - t\sin t), \tag{2.5}$$
$$\Im(w) = e^s(s\sin t + t\cos t). \tag{2.6}$$

While $\exp(s + it) = \exp(s)\exp(it)$ was periodic in t with period 2π, w is not. Then how to proceed? We can do the reverse of what we did in the case of the logarithm. There we first realized that horizontal lines of distance 2π are mapped to the same half-line by exp, and we selected the half-line with $-\pi$ angle (for matching with the tradition and with the usual definition of the arg function). Now let us see how the negative real axis is mapped by the Lambert function. If they partition the range of W, we are doing fine.

The complex number w falls onto the negative real axis if $\Im(w) = 0$, and $\Re(w) < 0$. The first requirement, by (2.6), means that

$$t = 0, \quad \text{or} \quad s\sin t + t\cos t = 0 \quad (\Longleftrightarrow \quad s = -t\cot(t)). \tag{2.7}$$

The inequality $\Re(w) < 0$ is equivalent with

$$s\cos t < t\sin t.$$

The equations in (2.7) are actually sufficient. If we draw the parametric curves encoded in (2.7), we see the plot on Figure 13: we have a partition of the complex plane by these curves.

We therefore have the visual information, how the set

$$\{W(u + iv) : u < 0,\ v = 0\}$$

partitions the z-plane. This suggests that our choice of making the *negative real line to be the branch cut* is (almost) correct. Why only "almost" will be seen in the next subsection.

2.5.2 The branch separating curves

The straight line separating W_{-1} and W_1

If $t = 0$, the situation is special. In this case $\Re(w) = se^s$, and we previously made the restriction $\Re(w) < 0$. This is satisfied when $s < 0$. But then the line

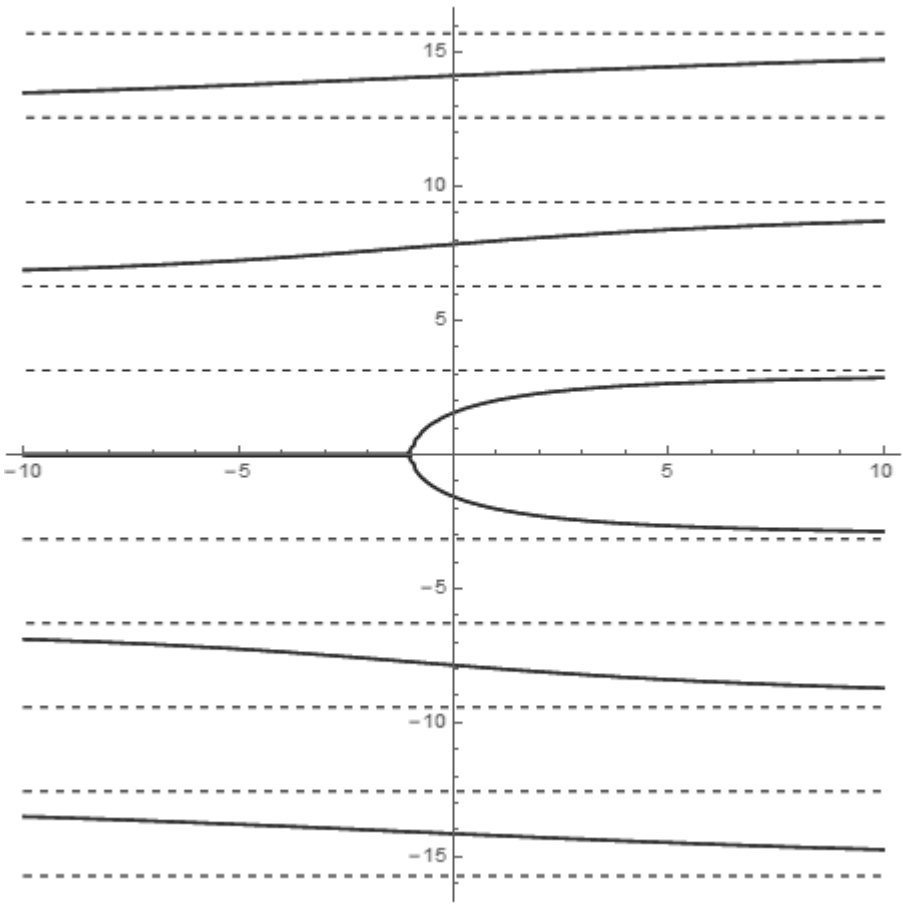


FIGURE 13
The partition of the z-plane by the images of the negative real axis in the range of the Lambert function (i.e., the inverse images of the negative real axis by $z \mapsto z\exp(z)$). The dashed horizontal lines have their ordinate at non-zero multiples of π. The thick curves are asymptotic to these.

segment $-1 \leq s < 0$ will belong entirely to the principal branch (see Figure 13, where already the correct separation is shown). Therefore, we must refine our idea a bit, and we have to choose not the whole negative real line, only the part for which $s \leq -1$. This line separates the W_{-1} and W_1 branches. In the w-plane, this corresponds to the *branch cut* $\mathbb{R} \ni w < (-1)e^{-1} = -1/e$. A *branch point* in the domain of W is therefore $w = -1/e$. These belong to W_{-1}, and W_0.

The other curves

The curves limiting the branches are more complicated than the simple horizontal lines of the logarithm, but their asymptotic behavior is easy to find. For these curves (if $t \neq 0$) it holds that

$$s = -t\frac{\cos(t)}{\sin(t)} = -t\cot(t). \tag{2.8}$$

When s tends to plus or minus infinity, then t must tend to the non-zero multiples of π from the corresponding direction.

The principal branch is limited by the curve

$$\{-t\cot(t) + it : -\pi < t < \pi\}.$$

(Notice that $-t\cot(t)$ has only a removable singularity at $t = 0$, where the limit is -1.)

The other curves are

$$\{-t\cot(t) + it : 2k\pi < \pm t < (2k+1)\pi\} \quad (k = 1, 2, \dots).$$

These all correspond to the negative real axis in the domain of W, so the negative real line is another *branch cut* for the Lambert function; and the origin is thus a *branch point*.

It is now seen that W has two branch cuts: the negative real line, which belongs to infinitely many branches, and the half-line $w < -1/e$ which belongs to W_{-1}, and W_0. The corresponding branch points are the origin, and $-1/e$, respectively.

These curves form a part of a class of curves studied around 2500 years ago, and they are called trisectrix of Hippias or quadratrix of Dinostratus.

2.5.3 The trisectrix of Hippias

The invention of the trisectrix is attributed to Hippias of Elis (5th century BCE), who used it to solve the *angle trisection problem* (hence the name *trisectrix*). About 70 years later, Dinostratus (390-320 BCE) used the same curve in the study of the problem of *squaring the circle*. This explains why the curve is also called *quadratrix*.

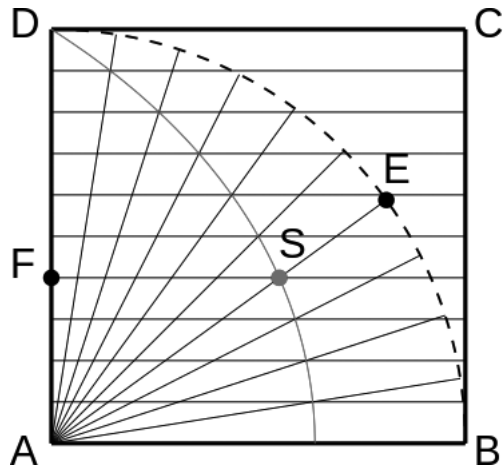


FIGURE 14
The construction of the quadratrix.

Both the angle trisection problem and squaring the circle is an impossible task if one only uses ruler and compass. However, if the quadratrix is at our disposal (that is, we have a quadratrix template or a quadratrix compass), both problems become solvable.

The description of the curve is as follows. Consider a square $ABCD$ as depicted in Figure 14, and draw an inner circle with center A connecting the points B and D. Now take a point E which travels with constant angular velocity on the circle; and take another point F which travels with constant velocity from point D downward to point A. The points E and F start moving from D at the same time instant, and the velocities are chosen such that the points will arrive at B and at A at the same time. The trisectrix/quadratrix is defined to be the set of points S which are the intersections of the horizontal line through F and the line segment $\overline{AE}$.

The quadratrix can be drawn by using the tool depicted in Figure 15.

It is clear from the construction how to trisect an angle, or even more generally how to construct the angle α/n if α is a fixed angle and $n \geq 2$. The traversed vertical line segment by the point F is proportional to the traversed arc by the point E, thus dividing the line segment into n equal parts (between A and F) we divide the angle $\angle EAB$ into n equal parts as well. The process for $n = 3$ is drawn on Figure 16.

The quadratrix can be given by a one-parameter curve. Place the square $ABCD$ on the Cartesian coordinate system such that A is placed at the origin, the vertex $\overline{AB}$ on the x-axis. Supposing that the length of the vertices of the

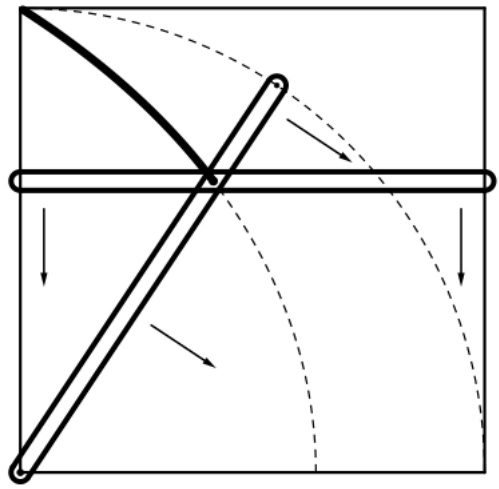


FIGURE 15
The quadratrix compass.

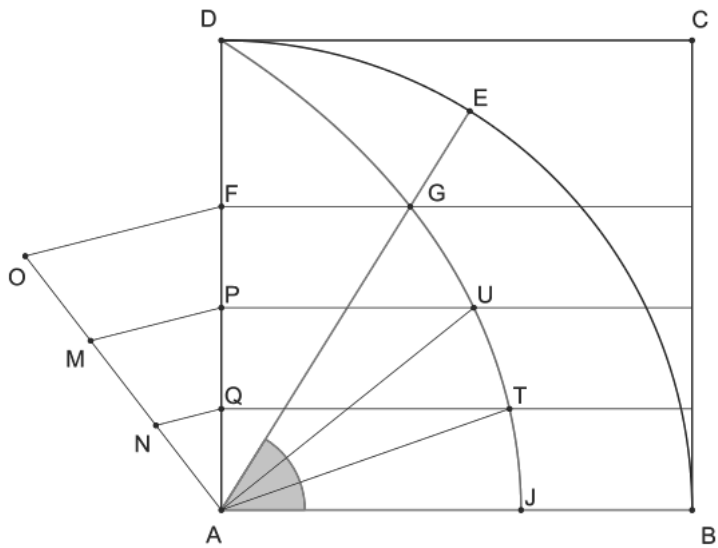


FIGURE 16
The solution of the angle trisection problem by using the trisectrix.

square is a, the quadratrix is given as

$$\frac{2a}{\pi}(t\cot(t), t), \quad t \in \left]0, \frac{\pi}{2}\right].$$

It is now apparent that the quadratrix indeed parametrizes the branch separating curves, see (2.8).

The three figures of this subsection were taken from *WikiMedia Commons*, see the article "Quadratrix of Hippias". For more on the history, the reader is directed to [71].

2.5.4 The closure of the branches

We still have to sort out how the branches are closed. Remember that in the case of the logarithm, we used the *counter-clockwise continuity* (CCC) principle to decide where the branches of the logarithm will be closed and open; see Figures 4 and 6.

CCC will be applied for the branches of W as well. What we need to do is find out the images of circles by W, one around the branch point $-1/e$, and another around the other branch point 0.

The branch point at $-1/e$

A circle of radius $r > 0$ around $-1/e$ in the w-plane satisfies the equation

$$\left(\Re(w) + \frac{1}{e}\right)^2 + \Im(w)^2 = r,$$

which, by (2.5)-(2.6), corresponds to the curve in the z-plane satisfying the equation

$$\left(e^s(s\cos t - t\sin t) + \frac{1}{e}\right)^2 + \left(e^s(s\sin t + t\cos t)\right)^2 = r.$$

Figure 17 shows this curve (where the radius is chosen to be one). By the CCC principle, the lower half of the limiting line therefore belongs to W_{-1} instead of W_0, while the upper half belongs to W_0. In other words, W_0 is "open from below" and "closed from above": see Figures 19-20.

The branch point at 0

A circle around the other branch point, 0, satisfies the equation

$$\left(\Re(w) + \frac{1}{e}\right)^2 + \Im(w)^2 = r,$$

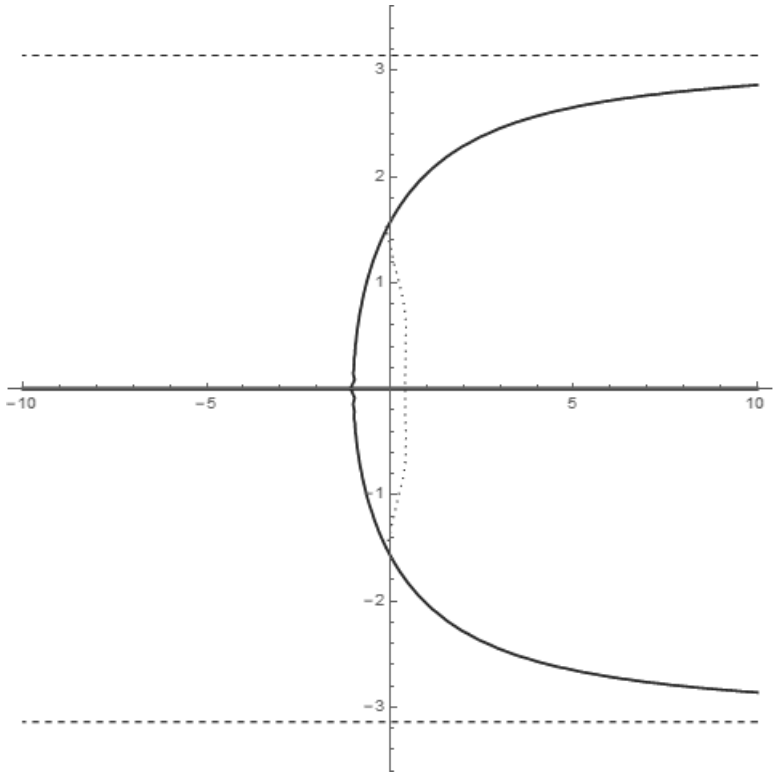


FIGURE 17
The image of a circle by the principal branch of the W function in the z-plane. By the CCC principle, the lower half of the branch limiting line therefore belongs to W_{-1}, the upper half belongs to W_0.

which, in the z-plane, gives the curve parametrized as

$$(e^s(s\cos t - t\sin t))^2 + (e^s(s\sin t + t\cos t))^2 = r,$$

as it is seen on Figure 18. Since the curve goes upward, we infer by the CCC principle that the branches are open from below, and closed from above. See Figures 19, 20, and 21, respectively, for the branches W_{-1}, W_0 and W_1.

This choice of the closure is in agreement with our previous agreement that W_{-1} is the other branch having real values. If the branch closures were chosen differently, W_1 would have this feature.

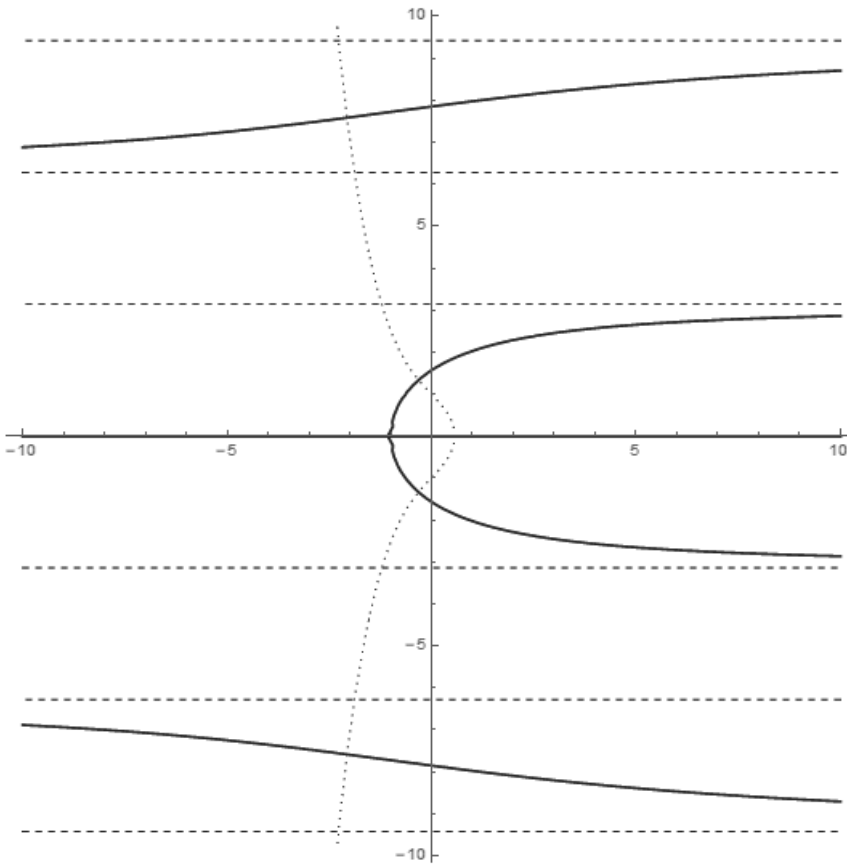


FIGURE 18
The image of a circle by the W function. By the CCC principle, the branches are open from below, and closed from above.

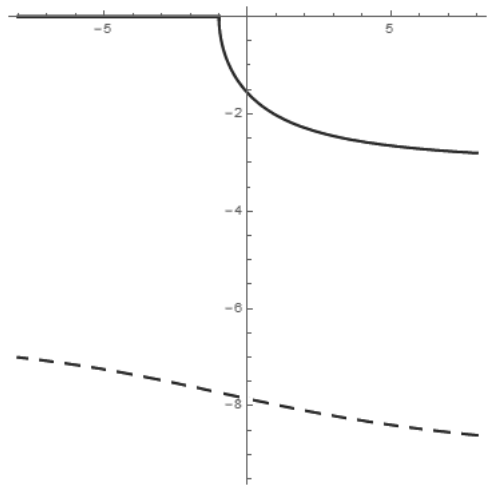


FIGURE 19
The closure of the branch W_{-1}.

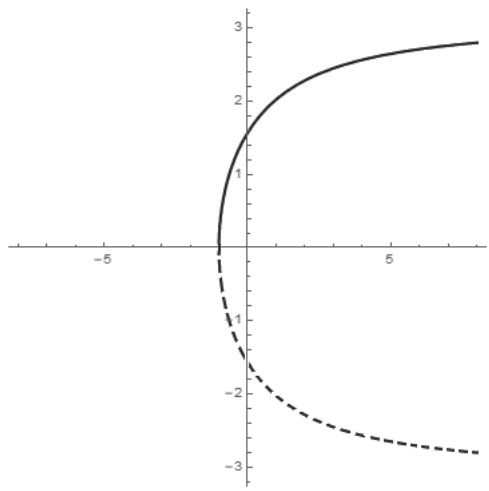


FIGURE 20
The closure of the branch W_0.

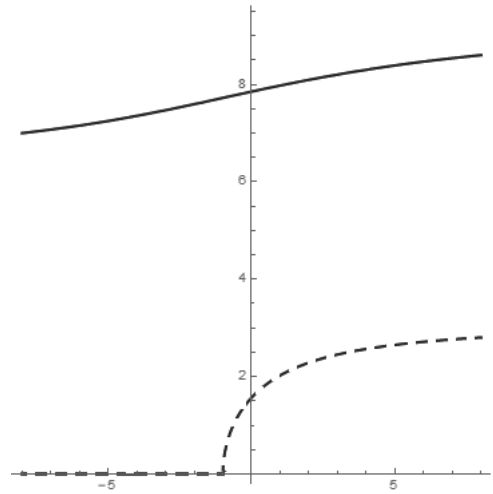


FIGURE 21
The closure of the branch W_1.

2.5.5 The Riemann surface of the Lambert function

Now we study the Riemann surface and monodromies of the Lambert W function.

We number the sheets according to the branches they represent. Thus, for example, the zeroth sheet will belong to W_0, and so on. The ranges of the principal branch and that of W_{-1} meet along a curve; see Figure 13. This curve belongs to W_{-1}, and it is the image of the interval $[-\infty, -1/e]$. If we want this connection continuous, we must cut up the sheets of W_0 and W_{-1} along the interval $[-\infty, -1/e]$, and glue them together. Because of how the closures are fixed, the interval itself still belongs to the sheet of W_{-1} when glued to the same interval on the sheet of W_0. All of this is shown in the corresponding part of Figure 22.

On the other hand, the sheet of W_{-1} still must somehow be glued to W_{-2} and W_1, because the ranges of these functions *meet* along curves. Let us deal with the connection of W_{-1} and W_{-2} first.

The curve along which W_{-1} and W_{-2} meet is the image of the negative real axis; therefore, we cut the -2nd sheet along $[-\infty, 0]$. The -1st sheet is already cut until $[-\infty, -1/e]$. We cut up more this sheet, from $]-1/e, 0]$. Next, we glue these two half-lines of the two sheets such that the half-line $[-\infty, 0]$ itself belongs to the -2nd sheet (again, this is because of the closure choice we made earlier).

Next, we connect the sheets of W_{-1} and W_1. The branch cut on the -1st sheet is almost entirely glued, only the interval $]-1/e, 0]$ is left. But this is just how it should be, because $W_{-1}(]-1/e, 0]) =]-\infty, -1[$, the line along

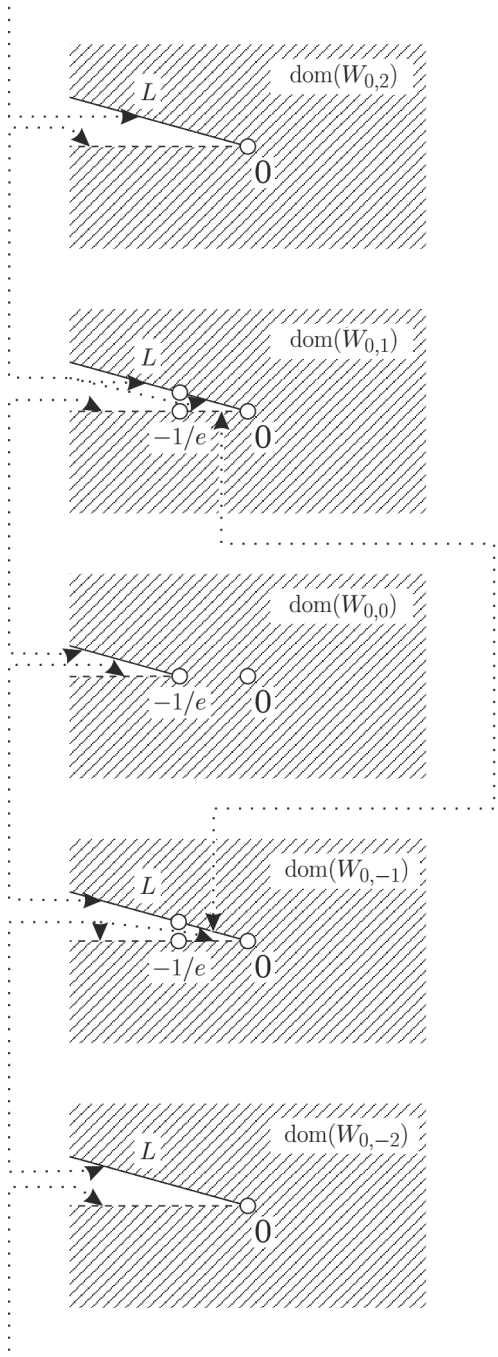


FIGURE 22
Gluing of the sheets of the Riemann surface of W. L is the half-line of non-positive reals.

which W_{-1} and W_1 meet! We therefore glue the interval $]-1/e, 0]$ on the two sheets in question. Note that this interval belongs to the -1st sheet when glued.

The zeroth and first sheets must also be glued together, because there is a curve between the corresponding branches of W. This curve is $W_0(]-\infty, -1/e[)$, so we take the interval $]-\infty, -1/e[$ on the two sheets and glue them together, such that the interval belongs to the zeroth sheet.

It is worth to check now the branch cuts of the first, zeroth, and minus first sheets: all of them have already been glued together in such a way that is consistent with the closures fixed previously by the CCC principle!

The rest is easy: we glue the branch cut of the -2nd sheet to the cut made on the -3rd sheet. By CCC, the interval belongs to the -3rd sheet. We go downwards similarly. Next, the first sheet is glued along its cut to the second sheet; the interval itself belongs to the first sheet. We then glue the sheets above these in the same way.

Notice one peculiarity: the -1st and first sheets are glued together, but this connection does not involve the zeroth sheet. This means that the Riemann surface of W cannot be drawn in the three-dimensional space without intersecting the zeroth sheet, because this latter is located "in between". This intersection is only virtual, if we embed the surface in a four-dimensional space, it has no intersections. This virtual intersection can perfectly be seen if we use the technique of Subsection 2.4.4, and draw the Riemann surface of W on a computer screen. This is done in Figure 23.

The *Mathematica* code for plot 23 is the following:

$$\begin{aligned}
&\text{ParametricPlot3D}\,[\{\exp(s)(s\cos(t)-t\sin(t)), \exp(s)(s\sin(t)+t\cos(t)), t\},\\
&\{t,-5,5\},\{s,-6,1\},\text{AspectRatio}\to 0.4,\\
&\text{ColorFunction}\to\text{Function}\,[\{x,y,z,t,s\},\\
&\quad\text{GrayLevel}\left[\frac{s+6}{7}\right]\Big],\text{ColorFunctionScaling}\to\text{False}\Big]
\end{aligned}$$

2.5.6 Monodromies

After seeing how the sheets of the Riemann surface of W are glued together, we have no difficulty when determining the monodromy groups of the surface encircling the two branch points $-1/e$ and 0.

Let us fix a circle of positive orientation around the origin of $\mathbb{C}$ with radius $< 1/e$, and with infinite winding number. This circle becomes an infinite open curve on the Riemann surface. Figure 22 shows us that if we follow the curve, it trespasses the sheets in this order:

$$\ldots, -3, -2, -1, 1, 2, 3, \ldots.$$

The zeroth sheet is not crossed at all. The monodromy group of the origin is the infinite cyclic group, $\mathbb{Z}$.

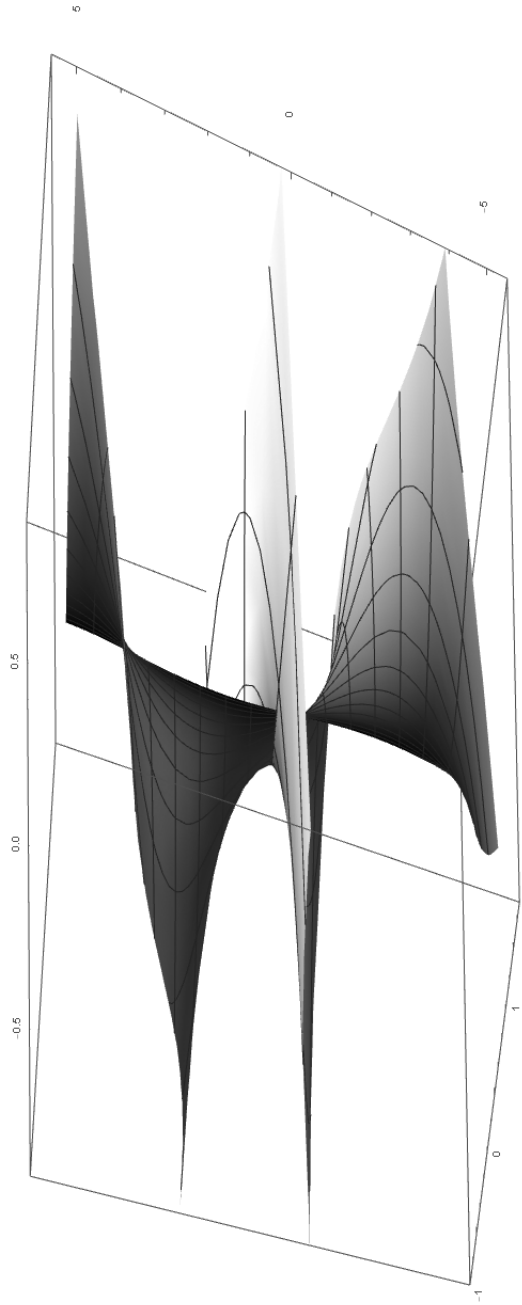


FIGURE 23
The Riemann surface of W.

Next, fix a circle around $-1/e$ with radius $< 1/e$ and of positive orientation. Now the winding number can be finite, because only three sheets (with index ± 1 and 0) will be involved. If we start on the zeroth sheet, we can easily follow the curve by following the way of gluing: $-1/e$ will be encircled entirely on the zeroth sheet starting on the dashed line of Figure 22. Reaching the solid line, we jump up to the first sheet, and we make a half-circle around $-1/e$ there, then we are led by the surface down to the minus first sheet, and make a half-circle there, too.

The monodromy group of the point $-1/e$ is therefore the cyclic group of order three, C_3.

To get a better visual information of how circles are mapped by the different branches of the Lambert function, we refer to Figures 24-25.

2.5.7 The fixed points of the Lambert function

The fixed points of the Lambert function are very easy to find, since these are the fixed points of its inverse ze^z, too. Since

$$ze^z = z, \quad \text{if and only if} \quad e^z = 1,$$

we see that

$$z = \log_k(1) = \log|1| + i(\arg(1) + 2k\pi) = 2k\pi i \quad (k \in \mathbb{Z}).$$

Thus $W(2k\pi i) = 2k\pi i$, but still we need to know which branch must be taken for the different k values. Thanks to the information that we have just gained about the branches of the Lambert function, a simple visual inspection of Figure 13 tells us that the branch index coincides with k. We get that

$$\boxed{W_k(2k\pi i) = 2k\pi i \quad (k \in \mathbb{Z}).}$$

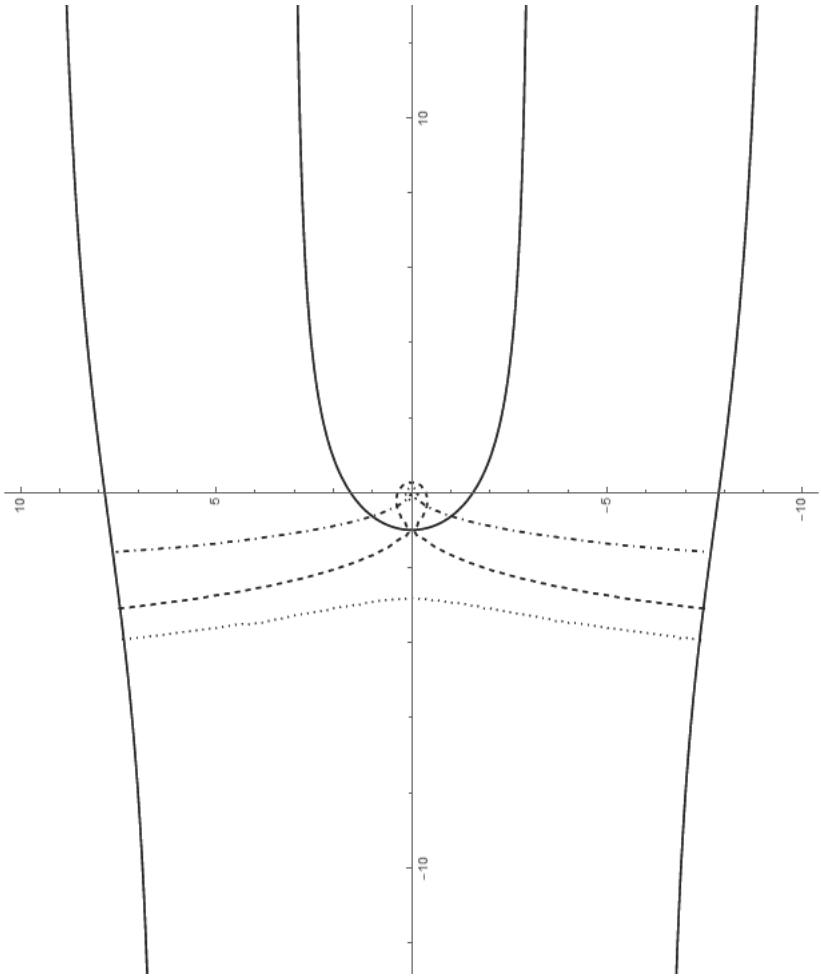


FIGURE 24
The images of three circles around the origin by W. The dotted curves (notice the tiny dotted curve around the origin inside the image of W_0) belong to the circle of radius $1/e - 0.2$. The dashed curves belong to the circle of radius $1/e$. The dotdashed curve is the image of the circle of radius $1/e + 1$.

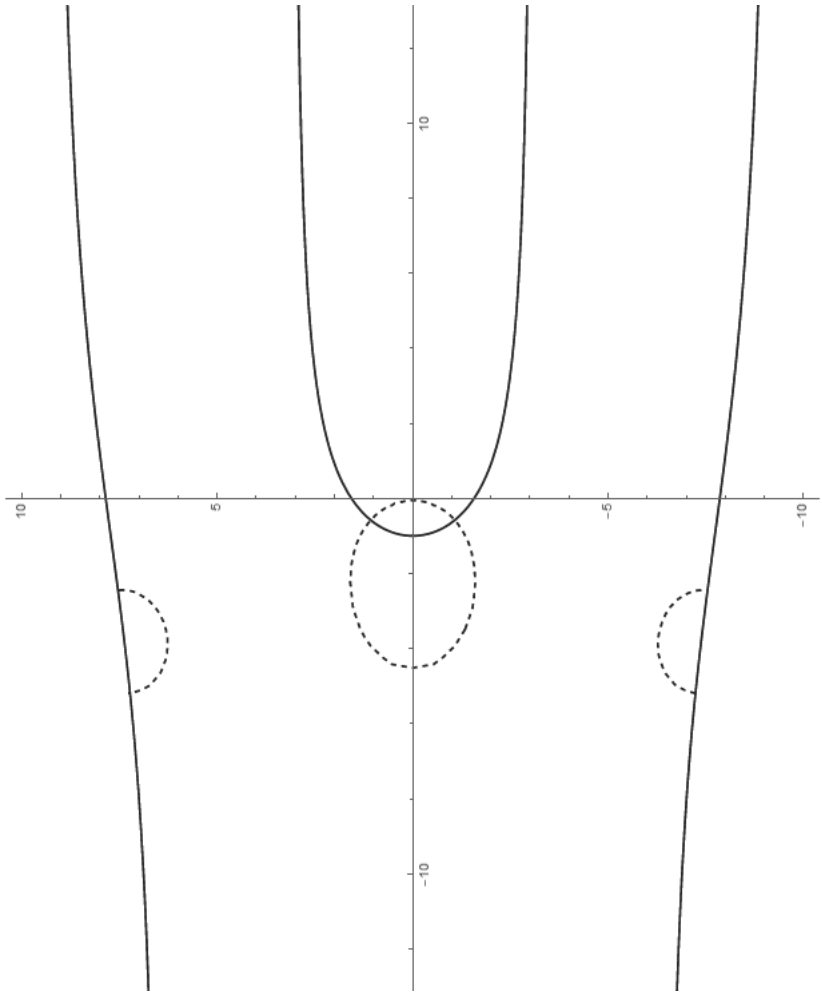


FIGURE 25
The images of three circles around $-1/e$ by W. The radius of the circle is $1/e - 0.05$.

Further notes

1. **Coverings.** The covering map is a very important notion in *algebraic topology*: it helps analyze the topological nature of the base space. It is also important in the study of algebraic curves. See more on this topic in [123].

2. **Monodromy group.** The definition of the monodromy group would require some preparation. The logarithm and even the Lambert function does not require such a heavy machinery, because their monodromy structure is very simple. So we do not need the exact definition of the monodromy group. The book [96] is suggested for those readers who would like to know more about the monodromy group in general. One can learn from this text, for example, that the monodromy group of $\sqrt{1+\sqrt{z}}$ is the dihedral group of order eight, a result which is not at all obvious.

3. **Some mapping properties of W.** By motivations coming from the analysis of delay differential equations, it was proven by Shinozaki [165] that $\Re(W_k(re^{j\theta}))$ is a monotone increasing function of r for any fixed non-zero k (Lemma A.3). Another useful fact proven by Shinozaki is that, outside of the branch cuts,

$$\max_{k\in\mathbb{Z}} Re(W_k(z)) = Re(W_0(z)).$$

3

Unwinding number, and branch differences. The functions $\mathcal{K}$, ω, and $\mathfrak{M}$

In this chapter, we make precise some sloppy "symbolic" calculations of the first chapter. It turns out that identities like $W(x) + \log W(x) = \log x$ (see (1.21)) are not true for arbitrary complex x, but some "correction terms" can be introduced such that the validity of these identities extends to the whole complex plane. This correction term is found via the so-called unwinding number $\mathcal{K}(z)$.

Second, it will be shown through an example, how to find *all* solutions of some equations from the first chapter. The Wright ω function, which is of independent interest, will be of great help in doing this.

In the last part of this chapter, we study the branch difference function $\mathfrak{M}_{m,n}(z) = W_m(z) - W_n(z)$. This branch difference is motivated by the branch difference of the logarithm. But, while $\ln_n(z) - \ln_m(z) = 2\pi(n-m)$, the function $\mathfrak{M}_{m,n}(z)$ is highly non-trivial and has some interesting properties.

3.1 Why does the complex version of many identities fail?

3.1.1 Non-identities regarding the logarithm

When solving logarithmic equations in high school, we frequently used identities like

$$\log(x^2) = 2\log x,$$

or

$$\log(xy) = \log x + \log y.$$

While these both are true when x and y are real, they fail to be true in general. Indeed, let us take $x = -1$ in the first displayed equation above. Then

$$\log((-1)^2) = \log 1 = 0,$$

but

$$2\log(-1) = 2(\log|-1| + i\arg(-1)) = 2\pi i. \tag{3.1}$$

DOI: 10.1201/9781003168102-3

This shows that the second identity also fails, because it is a generalized version of the first one; but it can be seen otherwise. Take $x = -1$, and $y = i$. Then

$$\log(xy) = \log(-i) = \log|-i| + i\arg(-i) = -\frac{\pi}{2}i.$$

On the other hand (by making use of (3.1)),

$$\log(x) + \log(y) = \log(-1) + \log(i) = \pi i + \log|i| + i\arg(i) = \pi i + i\frac{\pi}{2} = \frac{3\pi}{2}i.$$

What goes wrong? We can see it by carefully analyzing the proof of the non-identity $\log(xy) = \log x + \log y$. A possible proof is as follows. Start with $xy = e^{\log x}e^{\log y}$, then note that $e^{\log x}e^{\log y} = e^{\log x + \log y}$. So far, everything is correct. Now take logarithm:

$$\log(xy) = \log e^{\log x + \log y} \stackrel{!}{=} \log x + \log y.$$

While it is true, that $e^{\log x} = x$, it is *not* true, that $\log(e^x) = x$, so the very last step is not correct[1]. Indeed, there are infinitely many different values of the logarithm; we must specify which one we are taking. If the wrong branch is picked, we lose the validity of the identity.

It is therefore very useful to find the exact form of the difference $z - \log(e^z)$. We will do this later, but first we show that many other identities that seemingly have no relation to the logarithm go wrong when extending from the real line to the complex plane.

3.1.2 Non-identities regarding roots

We do not need to use the logarithm to see some confusing aspects of the "usual high school algebra" when extended to complex numbers.

It is seemingly okay to do the following calculation:

$$\sqrt{z-1}\sqrt{z+1} = \sqrt{(z-1)(z+1)} = \sqrt{z^2-1}. \tag{3.2}$$

Using this "identity", we infer, by putting $z = -2$, that

$$\sqrt{-3}\sqrt{-1} = \sqrt{3}. \tag{3.3}$$

On the left-hand side, we might argue that

$$\sqrt{-3}\sqrt{-1} = \sqrt{-1}\sqrt{3}\sqrt{-1}\sqrt{1} = i^2\sqrt{3} = -\sqrt{3}. \tag{3.4}$$

We have a sign mismatch here. Should we, instead, do the following?

$$\sqrt{-3}\sqrt{-1} = \sqrt{-1}\sqrt{3}\sqrt{-1}\sqrt{1} = \sqrt{(-1)(-1)}\sqrt{3} = \sqrt{3}. \tag{3.5}$$

[1]Take $x = 2\pi i$: $\log(e^{2\pi i}) = \log(1) = 0 \neq 2\pi i$. Instead, what is true is $\log(e^{2\pi i}) =$

This gives the answer which matches with (3.3). What is going on here? The problem is that at several points in the above calculations we used non-permitted branch choices. For example, if we had used $-\sqrt{-1}$ (which is the other branch of the square root) in (3.5) in one and only one place, we would have gotten $-\sqrt{3}$, as in (3.4).

If we want to do the calculations correctly, we must recall that, by definition,

$$\sqrt{z} = \exp\left(\frac{1}{2}\log z\right).$$

Here the principal branches are used for both the square root and the logarithm. Then

$$\sqrt{-3} = \exp\left(\frac{1}{2}\log(-3)\right) = \exp\left(\frac{1}{2}(\log 3 + i\arg(-3))\right) =$$

$$\exp\left(\frac{1}{2}\log 3 + \frac{\pi i}{2}\right) = \exp\left(\frac{1}{2}\log 3\right)\exp\left(\frac{\pi i}{2}\right) =$$

$$\sqrt{3}\exp\left(\frac{1}{2}\log(-1)\right) = \sqrt{3}\sqrt{-1} = i\sqrt{3}.$$

Thus $\sqrt{-3}\sqrt{-1} = i^2\sqrt{3} = -\sqrt{3} \neq \sqrt{3}$, which means that (3.2) is plainly incorrect. In general, it is false that $\sqrt{z_1 z_2} = \sqrt{z_1}\sqrt{z_2}$.

Again, we can see more closely where the problem appears by trying to do the proof:

$$\sqrt{z_1 z_2} = \exp\left(\frac{1}{2}\log(z_1 z_2)\right) \stackrel{!}{=} \exp\left(\frac{1}{2}\log z_1 + \frac{1}{2}\log z_2)\right) =$$

$$\exp\left(\frac{1}{2}\log z_1\right)\exp\left(\frac{1}{2}\log z_2\right) = \sqrt{z_1}\sqrt{z_2}.$$

We saw in the last subsection that $\log(z_1 z_2) = \log z_1 + \log z_2$ does not always hold, and we also saw that this has something to do with the fact that $\log(e^z) \neq z$.

We therefore now start to study the function $z - \log(e^z)$.

3.2 The unwinding number

3.2.1 The definition of the unwinding number

It is not always true that $z - \log(e^z) = 0$, and this difference manifests itself when we try to extend identities from the positive real line to the complex plane (and even when extending identities to negative reals).

Let $z = s + it$, and do the following calculation (with respect to the principal branch of log):

$$z - \log(e^z) = s + it - \log(e^{s+it}) = s + it - \left(\log|e^{s+it}| + i\arg(e^{s+it})\right).$$

Notice that $|e^{s+it}| = e^s$, and thus $\log|e^{s+it}| = \log|e^s| = \log e^s = s$ for positive real s. Moreover, $\arg(e^{s+it}) = \arg(e^{it})$. Since the argument is in $]-\pi, \pi]$, t and $\arg(e^{it})$ differs in an integer multiple of 2π: $t = \arg(e^{it}) + 2k\pi$ for some integer k. The above displayed line therefore gives that

$$z - \log(e^z) = it - i\arg(e^{it}) =$$

$$i(\arg(e^{it}) + 2k\pi) - i\arg(e^{it}) = 2k\pi i.$$

Thus the difference between z and $\log(e^z)$ is an integer multiple of $2\pi i$, which is not a surprise: this is the difference between the corresponding points on different branches of the logarithm.

Hence

$$k = \frac{z - \log(e^z)}{2\pi i}.$$

On the other hand, from the usual definition of the logarithm (see (2.3)) it is immediate that

$$k = \left\lceil \frac{\Im(z) - \pi}{2\pi} \right\rceil.$$

Indeed, notice that if $-\pi < \Im(z) \leq \pi$, then $k = 0$, which means that z lies on the horizontal strip of the principal branch. If $-3\pi < \Im(z) \leq -\pi$, then $k = -1$, which again correctly gives that we are on the branch of $\log_{-1}$. We therefore introduce the *unwinding number* as

$$\boxed{\mathcal{K}(z) : \mathbb{C} \to \mathbb{Z}; \quad \mathcal{K}(z) = \left\lceil \frac{\Im(z) - \pi}{2\pi} \right\rceil.} \tag{3.6}$$

With this function we have the universally true expression

$$\boxed{\log(e^z) = z - 2\pi i\mathcal{K}(z) \quad (z \in \mathbb{C}).} \tag{3.7}$$

The $\mathcal{K}(z)$ function (with a different sign) was introduced in the paper [41]. Later the authors noted [43] that calculations are easier (less minus signs appear) if $\mathcal{K}(z)$ is defined as we did here.

Let us see how the unwinding number corrects the wrong identities we dealt with previously.

$\log_1(1) = 2\pi i$, where $\log_1$ is the $k = 1$ branch of the logarithm.

3.2.2 The use of $\mathcal{K}(z)$

Until the end of this chapter, we will not need the notion of w- and z-planes; the arguments will be denoted by z, z_1, z_2, as usual.

Correcting $\log(z_1 z_2) = \log z_1 + \log z_2$

The non-identity $\log(z_1 z_2) = \log z_1 + \log z_2$ can be corrected by the use of $\mathcal{K}(z)$. This is done as follows. We start with

$$z_1 z_2 = \exp(\log z_1)\exp(\log z_2) = \exp(\log z_1 + \log z_2).$$

We should not be afraid; it is always true that $\exp(z_1)\exp(z_2) = \exp(z_1 + z_2)$. Now take (the principal branch of the) logarithm of both sides in the last displayed equation:

$$\log(z_1 z_2) = \log\exp(\log z_1 + \log z_2).$$

Here is the point where the unwinding number comes into the picture: the log of an exp must be corrected by $\mathcal{K}$:

$$\log\exp(\log z_1 + \log z_2) = \log z_1 + \log z_2 - 2\pi i\mathcal{K}(\log z_1 + \log z_2),$$

whence

$$\boxed{\log(z_1 z_2) = \log z_1 + \log z_2 - 2\pi i\mathcal{K}(\log z_1 + \log z_2).} \tag{3.8}$$

Applying this to our example on p. 71, we get that

$$\log(-i) = \log(-1) + \log(i) - 2\pi i\mathcal{K}(\log(-1) + \log(i)) =$$

$$\pi i + \frac{\pi}{2}i - 2\pi i\mathcal{K}\left(\frac{3\pi}{2}i\right) = \frac{3\pi}{2}i - 2\pi i\left\lceil\frac{\frac{3\pi}{2} - \pi}{2\pi}\right\rceil =$$

$$\frac{3\pi}{2}i - 2\pi i\left\lceil\frac{1}{4}\right\rceil = \frac{3\pi}{2}i - 2\pi i = -\frac{\pi}{2}i.$$

It can similarly be seen that

$$\boxed{\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2 - 2\pi i\mathcal{K}(\log z_1 - \log z_2),}$$

in particular,

$$\boxed{\log\left(\frac{1}{z}\right) = -\log z - 2\pi i\mathcal{K}(-\log z).} \tag{3.9}$$

Correcting $\sqrt{z_1 z_2} = \sqrt{z_1}\sqrt{z_2}$

We saw that $\sqrt{z_1 z_2}$ does not always equal $\sqrt{z_1}\sqrt{z_2}$, because the two branches of the square root mix things up. Since the two branches of the

square root only differ by a sign, we expect an occasional -1 factor multiplying $\sqrt{z_1}\sqrt{z_2}$.

The following chain of reasoning results in the correct answer:

$$\sqrt{z_1 z_2} = \exp\left(\frac{1}{2}\log(z_1 z_2)\right) =$$

$$\exp\left(\frac{1}{2}\log z_1 + \frac{1}{2}\log z_2 - \frac{2\pi i}{2}\mathcal{K}(\log z_1 + \log z_2)\right) =$$

$$\exp\left(\frac{1}{2}\log z_1\right)\exp\left(\frac{1}{2}\log z_2\right)\exp\left(-\pi i\mathcal{K}(\log z_1 + \log z_2)\right) =$$

$$\sqrt{z_1}\sqrt{z_2}\exp\left(-\pi i\mathcal{K}(\log z_1 + \log z_2)\right).$$

Since the unwinding number is integer, and $\exp(-\pi i k) = (-1)^k$, we get that

$$\boxed{\sqrt{z_1 z_2} = \sqrt{z_1}\sqrt{z_2}(-1)^{\mathcal{K}(\log z_1 + \log z_2)}.}$$

3.3 Corrected identities for the Lambert function

After seeing elementary examples which show that care must be taken when proving formulas with complex functions, we turn to the Lambert function. In the first chapter we saw numerous identities satisfied by W. While they are true for the principal branch at positive real argument (where everything is real and even positive), there are troubles when we try to extend those identities to other branches and complex arguments. In this section we clarify this issue, and the unwinding number will be very handy in doing so.

3.3.1 Correcting (1.21)

Before we start, let us prove a simple identity which will be needed:

$$\boxed{\mathcal{K}(z - 2\pi i n) = \mathcal{K}(z) - n.} \tag{3.10}$$

By the definition (3.6),

$$\mathcal{K}(z - 2\pi i n) = \left\lceil \frac{\Im(z - 2\pi i n) - \pi}{2\pi} \right\rceil = \left\lceil \frac{\Im(z) - 2\pi n - \pi}{2\pi} \right\rceil =$$

$$\left\lceil \frac{\Im(z) - \pi}{2\pi} - n \right\rceil = \mathcal{K}(z) - n.$$

To find the form of (1.21) which would be valid on the whole plane, we take the logarithm of $W_k(w)e^{W_k(w)} = w$ *correctly*. Leaving the argument of W_k, we get from (3.8) that

$$\log W_k + \log e^{W_k} - 2\pi i\mathcal{K}(\log W_k + \log e^{W_k}) = \log w.$$

Now, applying (3.7) for the second term, we have that

$$\log W_k + W_k - 2\pi i\mathcal{K}(W_k) - 2\pi i\mathcal{K}(\log W_k + \log e^{W_k}) = \log w. \tag{3.11}$$

The term $\log e^{W_k}$ inside the unwinding number is expanded again with (3.7):

$$\mathcal{K}(\log W_k + \log e^{W_k}) = \mathcal{K}(\log W_k + W_k - 2\pi i\mathcal{K}(W_k)).$$

Since $\mathcal{K}(W_k)$ is an integer, we can use (3.10) to see that

$$\mathcal{K}(\log W_k + W_k - 2\pi i\mathcal{K}(W_k)) = \mathcal{K}(\log W_k + W_k) - \mathcal{K}(W_k).$$

With this, (3.11) simplifies to

$$\log W_k + W_k - 2\pi i\mathcal{K}(\log W_k + W_k) = \log w. \tag{3.12}$$

It turns out that the unwinding number of $\log W_k + W_k$ is a simple function of k:

$$\mathcal{K}(\log W_{-1} + W_{-1}) = \begin{cases} 0, & \text{if } -\frac{1}{e} \le w < 0; \\ -1, & \text{otherwise,} \end{cases} \tag{3.13}$$

and if $k \neq 1$, then

$$\mathcal{K}(\log W_k + W_k) = k \tag{3.14}$$

Let us prove these identities.

Notice that $\Im(\log W_k) = \arg W_k$, and thus

$$\mathcal{K}(\log W_k + W_k) = \left\lceil \frac{\arg W_k + \Im(W_k) - \pi}{2\pi} \right\rceil.$$

This equals k when $\arg W_k + \Im(W_k) \in](2k-1)\pi, (2k+1)\pi]$. Recall (2.5)-(2.6). Fix $w = re^{i\theta}$, with which equations (2.5)-(2.6) become

$$\begin{aligned} r\cos\theta &= e^s(s\cos t - t\sin t), \\ r\sin\theta &= e^s(s\sin t + t\cos t). \end{aligned}$$

Divide the first equation by the second one (given that $\theta \notin \{0, \pi\}$):

$$\cot\theta = \frac{\cot(\arg W_k)\cot t - 1}{\cot(\arg W_k) + \cot t} = \cot(\arg W_k + \Im(W_k)).$$

These equalities hold for arbitrary r, and, in particular, also when $r \to \infty$. When w is large so that r is large but θ fixed (so we are moving on a ray R of the w-plane), $w = ze^z$ must be large, and z must be large, too. But for large z, the exponential dominates, and the inversion of ze^z is therefore close to the logarithm. In other words, the Lambert function asymptotically behaves like the logarithm. Since the ray R is mapped to a horizontal line by the logarithm, and points on R with large magnitude are mapped to points with large positive real part but fixed imaginary part, we see that $\lim_{r\to\infty} \arg\log R = 0$, the same is true for the branches of the Lambert function:

$$\lim_{r\to\infty} \arg W_k(re^{i\theta}) = 0 \quad (\theta \in]-\pi, \pi[\setminus \{0\}).$$

Consequently,

$$\lim_{r\to\infty} (\arg W_k + \Im(W_k)) = \lim_{r\to\infty} \Im(W_k) = \lim_{r\to\infty} t = \theta + 2k\pi.$$

This is so, because the image of R by the branches of W_k approach the horizontal lines of $\log(R)$ asymptotically, and $\Im \log_k(R) = \theta + 2k\pi$. It now follows, that $\arg W_k + \Im(W_k) \in](2k-1)\pi, (2k+1)\pi]$, except at least when $k = -1$. This case needs special attention because if $-1/e \le w < 0$, then $W_{-1}(w)$ is a negative real number, and

$$\arg W_{-1} + \Im(W_{-1}) = \arg W_{-1} = \pi,$$

so now $\mathcal{K}(\log W_{-1} + W_{-1}) = 0$, as stated.

Using (3.13)-(3.14) with (3.12), we get that

$$\boxed{W_k(z) + \log W_k(z) = \begin{cases} \log z, & \text{if } k = -1,\ z \text{ is real,} \\ & \text{and } -1/e \le z < 0; \\ \log z + 2\pi i k, & \text{otherwise.} \end{cases}} \tag{3.15}$$

3.3.2 Correcting (1.23)

The neat formula (1.23) holds only for real values, but in this section we extend it to the whole complex plane. Our statement is the following.

For all $a, b, z, w \in \mathbb{C}$, and integer h, we have that

$$aW_h(z) + bW_h(w) =$$

$$W_k\left[(aW_h(z)+bW_h(w)) \left(\frac{z}{W_h(z)}\right)^a \left(\frac{w}{W_h(w)}\right)^b \exp\left(2\pi i[a\mathcal{K}(W_h(z))+b\mathcal{K}(W_h(w))]\right)\right],$$

where the branch index k must be chosen as follows. Let $g = aW_h(z)+bW_h(w)$, and $f = -\Im(g)\cot(\Im(g))$.

1. If $\Im(g) = 0$, and $\Re(g) \ge -1 \Rightarrow k = 0$,

2. if $\Im(g) = 0$, and $\Re(g) < -1$, $\Rightarrow k = -1$,
3. if $0 < \Im(g) < \pi$, and $\Re(g) \geq f \Rightarrow k = 0$,
4. if $0 < \Im(g) < \pi$, and $\Re(g) < f \Rightarrow k = 1$,
5. if $-\pi < \Im(g) < 0$, and $\Re(g) \leq f \Rightarrow k = -1$,
6. if $-\pi < \Im(g) < 0$, and $\Re(g) > f \Rightarrow k = 0$,
7. if $\Im(g) = n\pi$ with odd $n > 0$, and $\Re(g) < +\infty \Rightarrow k = \frac{n+1}{2}$, and if $\Re(g) = +\infty \Rightarrow k = \frac{n-1}{2}$,
8. if $\Im(g) = n\pi$ with even $n > 0 \Rightarrow k = n/2$,
9. if $\Im(g) = n\pi$ with even $n < 0$, and $\Re(g) > -\infty \Rightarrow k = n/2$,
10. if $\Im(g) = n\pi$ with even $n < 0$, and $\Re(g) = -\infty \Rightarrow k = n/2 - 1$,
11. if $\Im(g) = n\pi$ with odd $n < 0 \Rightarrow k = \frac{n-1}{2}$.

Moreover, if $\Im(g)$ is not a multiple of π, and it is in the complement of the interval $[-\pi, \pi]$, then let $n := \lfloor \Im(g)/\pi \rfloor$. Now,

12. if $\Im(g) > 0$, n is odd $\Rightarrow k = \frac{n+1}{2}$,
13. if $\Im(g) > 0$, n is even, and $\Re(g) < f \Rightarrow k = n/2 + 1$,
14. if $\Im(g) > 0$, n is even, and $\Re(g) \geq f \Rightarrow k = n/2$,
15. if $\Im(g) < 0$, n is even $\Rightarrow k = n/2$,
16. if $\Im(g) < 0$, n is odd, and $\Re(g) \leq f \Rightarrow k = \frac{n-1}{2}$,
17. if $\Im(g) < 0$, n is odd, and $\Re(g) > f \Rightarrow k = \frac{n+1}{2}$.

The proof

We look for an expression for $aW_h(z) + bW_h(w)$, so we write

$$[aW_h(z) + bW_h(w)]e^{aW_h(z)+bW_h(w)} = s(z, w, a, b, h) =: s.$$

The solution of this equation is, by definition, $W_k(s)$ with a given branch index k, such that s falls into the image of $W_k(s)$. This requirement will be dealt with later.

For the moment, we have that

$$W_k(s) = aW_h(z) + bW_h(w). \tag{3.16}$$

We rewrite s as

$$s = [aW_h(z) + bW_h(w)]e^{aW_h(z)+bW_h(w)} =$$

$$[aW_h(z) + bW_h(w)]e^{aW_h(z)}e^{bW_h(w)}.$$

Recall that $e^{W_h(z)} = z/W_h(z)$, and $e^{W_h(w)} = w/W_h(w)$. In the next step, we elevate these to the powers a and b, respectively. Here we must be careful, since it does not hold true in general, that $e^{ba} = (e^b)^a$. Instead, what is true is

$$e^{ba} = (e^b)^a e^{2\pi i a\mathcal{K}(a)},$$

which is a particular case of the more general identity

$$z^{ba} = (z^b)^a e^{2\pi i a\mathcal{K}(a\log z)},$$

see [25].

Now we therefore have that

$$e^{aW_h(z)} = (e^{W_h(z)})^a e^{2\pi i a\mathcal{K}(W_h(z))} = \left(\frac{z}{W_h(z)}\right)^a e^{2\pi i a\mathcal{K}(W_h(z))},$$

and, similarly,

$$e^{bW_h(w)} = \left(\frac{w}{W_h(w)}\right)^b e^{2\pi i b\mathcal{K}(W_h(w))}.$$

Upon substituting these into the expression of s, we see that

$$s = [aW_h(z) + bW_h(w)]\left(\frac{z}{W_h(z)}\right)^a \left(\frac{w}{W_h(w)}\right)^b e^{2\pi i[a\mathcal{K}(W_h(z))+b\mathcal{K}(W_h(w))]}.$$

This and (3.16) yield the formula in our statement.

We still need to find the proper branch index k. Since the determination of k is elementary but laborious, we will consider only three particular, but representative, cases.

1. Let us suppose that $g = aW_h(z)+bW_h(w)$ is real. The only branches which contain real values belong to the indices $k = 0$ and $k = -1$. If $k = 0$, the range of the function W_0 contains $[-1, +\infty[$, and the range of W_1 contains $]-\infty, -1]$. Consequently, if $\Re(g) \geq -1$ we have that $k = 0$, otherwise $k = -1$. This shows the first two particular cases in our theorem. (Note that if $\Re(g) = -1$, we could equivalently choose $k = -1$.)

2. Next, let us suppose that $\Im(g)$ is a positive multiple of π. First, let n be odd. By inspecting Figure 13, we see that all the points $\xi + n\pi i$ belong to the branch $k = (n + 1)/2$ for finite ξ. However, because of the branch closure, of $\xi = +\infty$, we are still on the branch $(n - 1)/2$. If n is even, the whole extended horizontal line $\xi + n\pi i$ is contained in the range of the branch indexed by $n/2$. These considerations cover the seventh and eighth cases in our theorem.

3. We now pick a case when $\Im(g)$ is positive, but it is not a multiple of π. If $\Im(g)$ is in the interval $]n\pi, (n+1)\pi[$ with even n (note that $n = \lfloor \Im(g)/\pi \rfloor$), then we have two possibilities: (1) $\Im(g)$ falls strictly on the left-hand side of the curve separating $W_{n/2}$ and $W_{n/2+1}$. The separating curve's real coordinate is $-\Im(g)\cot(\Im(g)) =: f$, and thus falling on the left of this curve means that $\Re(g) < f$; in which case $k = n/2 + 1$ (confirming the case 13). (2) $\Im(g) \geq f$. Since the curve itself, and the points with real coordinate greater than f all fall into the range of $W_{n/2}$, we get that $k = n/2$ (this is case 14). Now let $\Im(g)$ belong to the interval $]n\pi, (n+1)\pi[$ with odd n. Since all the finite points with such an imaginary coordinate belong to $W_{(n+1)/2}$, we get that $k = (n+1)/2$. This is case 12 in our theorem.

The rest of the cases can be seen similarly.

Some neat examples

To see how the theorem is applied, we deduce several special values for the Lambert function.

For example, the following special value for W_{-1} is given:

$$\boxed{W_{-1}\left(\frac{\Omega - e}{\Omega e^e}\right) = \Omega - e.}$$

Choose $a = 1$, $b = -1$, $z = 1$, $w = e \cdot e^e$, and $h = 0$. Then

$$aW_h(z) + bW_h(w) = W_0(1) - W_0(e \cdot e^e) = \Omega - e.$$

Since $\Omega - e \approx -2.15114$, we get that $k = -1$ (see case 2), and our formula states that

$$\Omega - e = W_{-1}\left((\Omega - e)\frac{1}{W_0(1)}\left(\frac{e \cdot e^e}{W(e \cdot e^e)}\right)^{-1} e^{2\pi i \cdot 0}\right) =$$

$$W_{-1}\left((\Omega - e)\frac{1}{\Omega}\left(\frac{e \cdot e^e}{e}\right)^{-1}\right) = W_{-1}\left(\frac{\Omega - e}{\Omega e^e}\right),$$

which confirms our result.

To see a complex special value, we present the following corollary.

$$\boxed{W_0\left(\frac{i\Omega}{\Omega^i}\right) = i\Omega.}$$

To see this, substitute $a = b = i/2$, $z = w = 1$, $h = 0$, such that

$$aW_h(z) + bW_h(w) = \frac{i}{2}W_0(1) + \frac{i}{2}W_0(1) = i\Omega.$$

The branch index is given by using case 3: $0 < \Im(i\Omega) = \Omega < \pi$, and

$$-\Im(i\Omega)\cot(\Im(i\Omega)) = -\Omega\cot(\Omega) \approx -0.89 < \Re(i\Omega) = 0,$$

so $k = 0$, indeed. (Note that the presented identity is the particular case of (1.29), but that was proven only for real x values.)

Similarly,

$$\boxed{W_0\left(\frac{2i\Omega}{\Omega^{2i}}\right) = 2i\Omega,}$$

but then a jump on the branch comes:

$$\boxed{W_1\left(\frac{3i\Omega}{\Omega^{3i}}\right) = 3i\Omega.}$$

Indeed,

$$-\Im(3i\Omega)\cot(\Im(3i\Omega)) = -3\Omega\cot(3\Omega) \approx 0.22 > \Re(3i\Omega) = 0,$$

thus $k = 1$, as case 4 tells us.

3.4 The Wright ω function

3.4.1 Looking at equation $x + \log x = a$ more carefully

The reader has seen a large number of examples on the previous pages where the usual high school algebra was shown to be failing. Suspicion therefore arises whether the solutions presented for the equations in Section 1.1 are indeed correct. If we are interested in *real* solutions (and this is the case in most of the applications), we can still do the usual algebraic manipulations to get *a* correct answer – as it was done in Section 1.1. However, a careful analysis is necessary if we want to find *all* the solutions of a given equation. In this section we scrutinize equation (1.13) (for $b = e$, and now writing z instead of a):

$$x + \log x = z. \tag{3.17}$$

We saw on p. 7 that $W(e^z)$ solves the equation. But we know now what we did not know when we first studied this equation: W has infinitely many branches. How do we take this into account to find *all* the solutions of (3.17)? Do we simply put a branch index and say that the solutions of $x + \log x = z$ are $W_k(e^z)$, where $k \in \mathbb{Z}$? The answer is a bold "no". While

$$W_0(e^3) + \log W_0(e^3) = 3,$$

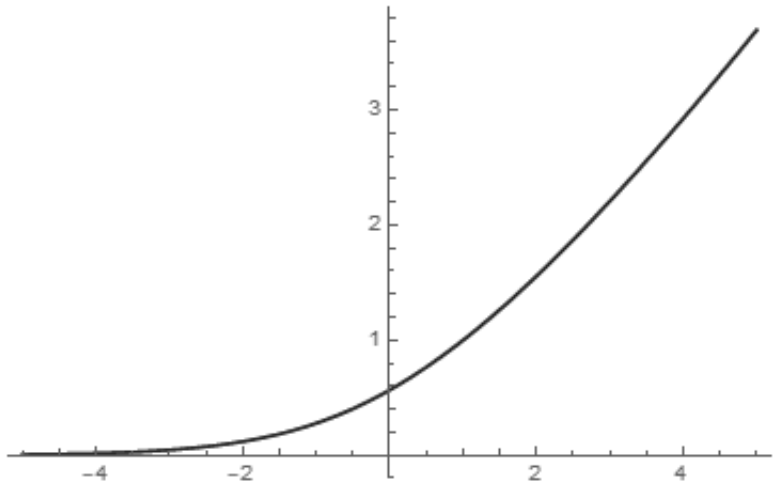


FIGURE 26
The plot of $\omega : \mathbb{R} \to]0, \infty[$.

it does not hold true that

$$W_{-1}(e^3) + \log W_{-1}(e^3) = 3, \quad \text{or} \quad W_1(e^3) + \log W_1(e^3) = 3.$$

It turns out that the branch index depends on z. To find the most elegant way of solving (3.17), we introduce the Wright ω function.

3.4.2 The definition and basic properties of ω

The Wright omega function was introduced in 2002 by Robert M. Corless and David J. Jeffrey [43]:

$$\boxed{\omega(z) = W_{\mathcal{K}(z)}(e^z).} \tag{3.18}$$

Notice that this function takes real values on $\mathbb{R}$ (see Figure 26). Because $\mathcal{K}(\mathbb{R}) = 0$, and $\exp(\mathbb{R}) =]0, \infty[$, we see that $\omega(\mathbb{R}) = W_0(]0, \infty[) =]0, \infty[$.

Moreover, ω is *single-valued*: for any fixed $z \in \mathbb{C}$, $\omega(z)$ is unique. This useful property of ω follows from the same property of exp, and $\mathcal{K}$.

ω solves $x + \log x = z$

To see how to solve (3.17), we first take the exponential of both sides of (3.17):

$$e^{x+\log x} = e^z \quad e^x \cdot x = e^z,$$

and therefore $x = W_k(e^z)$ is a solution for some branch(es) of W. We shall show that the solution (if exists) is unique, except on a half-line. There is another half-line on which there is no solution at all.

Firstly, recall (3.15). Replace z with e^z in the second part of this formula:

$$W_k(e^z) + \log W_k(e^z) = \log(e^z) + 2\pi i k.$$

We are well aware that $\log(e^z)$ is not z in general; instead, $\log(e^z) = z - 2\pi i\mathcal{K}(z)$ holds true (see (3.7)). Hence

$$W_k(e^z) + \log W_k(e^z) = z + 2\pi i(k - \mathcal{K}(z)), \tag{3.19}$$

except when $k = -1$, and $-1/e \le e^z < 0$. Since $x = W_k(e^z)$ for some k, and $x + \log x = y$, it must hold true that $W_k(e^z) + \log W_k(e^z) = z$. According to (3.19), this is only possible in the non-exceptional case, when $k = \mathcal{K}(z)$. Thus, $W_{\mathcal{K}(z)}(e^z) = \omega(z)$ is a solution. This finding tells why the definition of ω is given as it is.

In the exceptional case, we instead have that

$$W_k(e^z) + \log W_k(e^z) = z - 2\pi i\mathcal{K}(z),$$

so $\mathcal{K}(z) = 0$ must hold. This amounts to another solution if z is a complex number such that $-1/e \le e^z < 0$, and $k = -1$. Let us see, what are the possible values of z in this case. Let $z = s + it$; then $e^z = e^s e^{it}$. If we want this number to be negative and real, $t = \pi + 2l\pi$ is necessary for some integer l. At the same time, $\mathcal{K}(z) = 0$ must hold, and this is only possible if $l = 0$. Remark, that if $l = -1$, the number z still satisfies the inequality $-1/e \le e^z < 0$, but for this z we have $\mathcal{K}(z) = -1 \neq 0$. Thus, on the set

$$L_0 = \{t - i\pi : -\infty < t \le -1\}$$

there is no solution of (3.17) at all.

Hence, $t = \pi$ is the only possible case, and $-1/e \le z = -e^s < 0$ holds only if $s \in]-\infty, -1]$. We have got that $W_{-1}(e^z)$ is a solution for any z for which $\Re(z) \in]-\infty, -1]$ and $\Im(z) = \pi$.

Notice, that for any such z, $W_0(e^z)$ is another solution. Indeed, for such z, $\mathcal{K}(z) = 0$, and the non-exceptional case is applied. Thus, there are *exactly two* solutions of (3.17) if z belongs to the line

$$L_2 = \{t + i\pi : -\infty < t \le -1\}.$$

Because of this fact, the line L_2 is called *doubling line*. For an illustration of L_0 and L_2, see Figure 27.

The two solutions on L_2 are $W_0(e^z)$ and $W_{-1}(e^z)$. The first one is just $W_0(e^z) = W_{\mathcal{K}(z)}(e^z) = \omega(z)$. We would like to express $W_{-1}(e^z)$ in terms of ω, too. We have, on L, that $W_{-1}(e^z) = W_{-1}(e^{z-2\pi i}) = W_{\mathcal{K}(z-2\pi i)}(e^{z-2\pi i}) = \omega(z - 2\pi i)$.

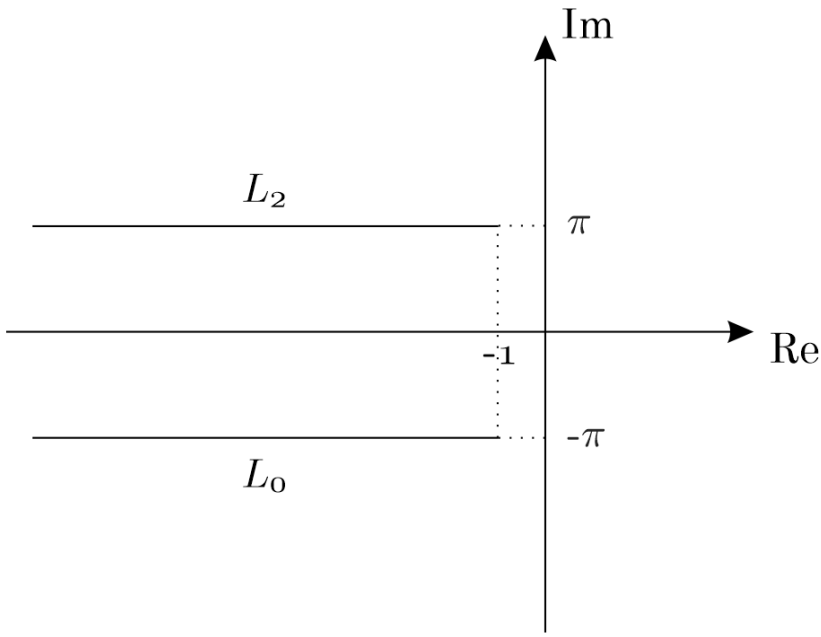


FIGURE 27
The lines L_0 and L_2.

Let us collect what we have deduced: the equation $x+\log x = z$ has exactly one solution if $z \in \mathbb{C} \setminus (L_0 \cup L_2)$, and this solution is $x = \omega(z)$. If $z \in L_0$, the equation has no solution at all. Finally, if $z \in L_2$, the equation has two solutions: $x_1 = \omega(z)$, and $x_2 = \omega(z - 2\pi i)$.

Expressing W in terms of ω

The omega function is defined in terms of W, but this relation can be reversed: the Lambert W can be calculated from ω in a nice and simple way. This is advantageous because the ω function is single-valued, while W is multi-valued. So, a simple parameterization of ω gives the value of W on an arbitrary branch.

We will prove the following statement:

$$\boxed{W_k(z) = \omega(\log(z) + 2\pi i k) \quad (z \in \mathbb{C}^*,\, k \in \mathbb{Z}).} \tag{3.20}$$

Here $\log(z)$ is meant to be the principal branch of the logarithm.

In fact, our presentation for the solutions of (3.17) already demonstrates the validity of our statement. If z is a non-zero complex number not belonging

to L_0 and L_2, it comes that, as we saw, $\omega(z) = W_k(e^z)$, and thus

$$\omega(\log z + 2\pi ik) = W_{\mathcal{K}(\log z + 2\pi ik)}(e^{\log z + 2\pi ik}) = W_k(z).$$

In the last step, we used the fact that log refers to the principal branch, so $-\pi < \arg(\log(z)) \leq \pi$, hence $\mathcal{K}(\log(z) + 2\pi ik) = k$.

3.4.3 The mapping properties of ω

There are some subtleties in the mapping properties of ω which are good to see in detail. The following facts are easy to see:

$$\begin{aligned}
\omega\left(\{e^z : z \in \mathbb{C},\, \Im(z) \in]-3\pi, -\pi]\}\right) &= W_{-1}(\mathbb{C}^*),\\
\omega\left(\{e^z : z \in \mathbb{C},\, \Im(z) \in]-\pi, \pi]\}\right) &= W_0(\mathbb{C}^*),\\
\omega\left(\{e^z : z \in \mathbb{C},\, \Im(z) \in]\pi, 3\pi]\}\right) &= W_1(\mathbb{C}^*),
\end{aligned}$$

etc.

This means that every horizontal strip of Figure 4 is mapped onto the corresponding area of Figure 13 (with the closures respected). In particular, ω maps $\mathbb{C}$ onto $\mathbb{C}^*$. Zero is excluded, because $W(z) = 0$ only when $z = 0$, but the exponential in its argument never takes the value zero.

The images of L_0 and L_2 by ω

We shall show now, that there is a value, which is taken twice by omega. In order to see this, we check the image of L_0 and L_2 by ω. On all the points of L_2, the unwinding number is zero,; therefore

$$\omega(L_2) = W_0(\exp(L_2)) = W_0\left(\left[-\frac{1}{e}, 0\right[\right) = [-1, 0[\,,$$

by (1.56). In particular,

$$\boxed{\omega(-1 + i\pi) = -1.}$$

Next, we determine $\omega(L_0)$. On L_0 the unwinding number is minus one everywhere, thus

$$\omega(L_0) = W_{-1}(\exp(L_0)) = W_{-1}\left(\left[-\frac{1}{e}, 0\right[\right) =]-\infty, -1].$$

This is seen from (1.55). It follows that

$$\boxed{\omega(-1 - i\pi) = -1.}$$

For a visual illustration of $\omega(L_0)$ and $\omega(L_2)$, see Figure 28.

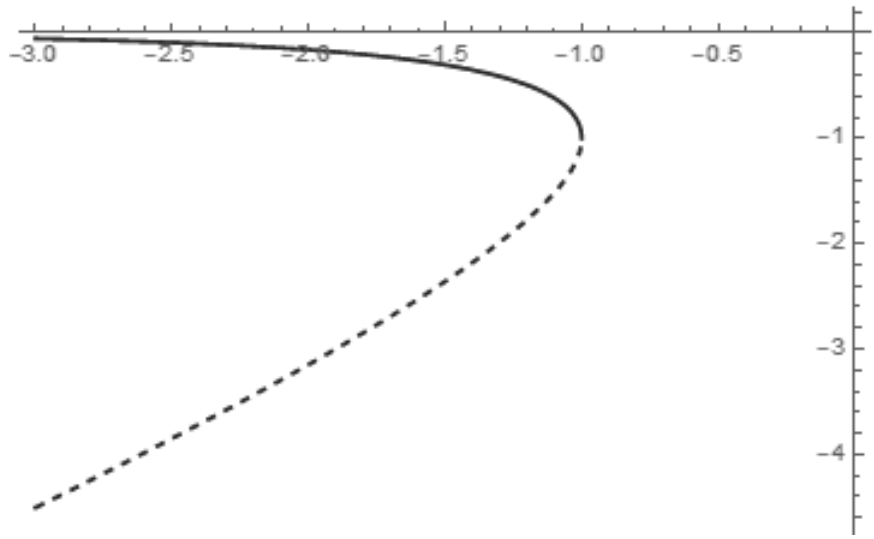


FIGURE 28
The plot of $\omega(L_2)$ (continuous plot) and $\omega(L_0)$ (dashed plot).

The existence of the doubling line causes that ω is not the straight functional inverse of $z + \log z$. Instead, the following result is valid:

$$\omega^{-1}(z) = \begin{cases} z + \ln z - 2\pi i & -\infty < z < -1 \\ -1 \pm i\pi & z = -1 \\ z + \ln z & \text{otherwise} \end{cases}$$

The proof of this fact is elementary, although a bit tedious because of the discontinuity of omega on the L_0 and L_2 lines. Regarding the continuity, we have the following result:

1. ω is analytic except at $z = t \pm i\pi$ for $t \leq -1$.
2. For $t \leq -1$, we have
   (a) $\omega\left(t + i\pi^{-}\right) = \omega(t + i\pi) = \omega\left(t - i\pi^{-}\right)$
   (b) $\omega\left(t + i\pi^{+}\right) = \omega(t - i\pi) = \omega\left(t - i\pi^{+}\right)$

We refer to [43] for the proofs. In the same paper, we can find the following table of special values of omega.

z	$\omega(z)$
$-\infty$	0
0	$W_0(1)$
1	1
$-1/3 + \ln(1/3) + i\pi$	$-1/3 = W_0\left(-\frac{1}{3}e^{-1/3}\right)$
$-1/3 + \ln(1/3) - i\pi$	$W_{-1}\left(-\frac{1}{3}e^{-1/3}\right)$
$-1 + i\pi$	-1
$-1 - i\pi$	-1
$-2 + \ln 2 + i\pi$	$W_0\left(-2e^{-2}\right)$
$-2 + \ln 2 - i\pi$	$-2 = W_{-1}\left(-2e^{-2}\right)$
∞	∞

3.5 The branch difference function $\mathfrak{M}_{m,n}$

The difference between two branches of the logarithm at the same point is a multiple of 2π. Indeed, for a fixed $z \in \mathbb{C}^*$

$$\log_m(z) - \log_n(z) =$$

$$\log|z| + i(\arg(z) + 2m\pi i) - \log|z| - i(\arg(z) + 2n\pi i) = 2\pi(m-n).$$

This observation leads to the question about the branch difference of the Lambert W function[2]. We define

$$\mathfrak{M}_{m,n}(z) = W_m(z) - W_n(z).$$

As the Lambert function satisfies the equation $W(z)e^{W(z)} = z$, the function $\mathfrak{M}_{m,n}(z)$ satisfies an equation, too. The elaboration of this statement will be given in the next subsection.

3.5.1 An equation for $\mathfrak{M}_{m,n}(z)$

Let $z \in \mathbb{C}$ be a fixed number, and let x be a variable. Consider the equation

$$\frac{x}{e^{-x}-1}\exp\left(\frac{x}{e^{-x}-1}\right) = z. \tag{3.21}$$

This equation has infinitely many solutions $x_{m,n}$, and these are given as

$$x_{m,n} = \mathfrak{M}_{m,n}(z) \quad (m,n \in \mathbb{Z}).$$

[2]The branch difference function was defined in 2014 by D. J. Jeffrey and J. E. Jankowski [91].

Let us see why the branch difference function solves this equation. First, observe that

$$\exp\left(\mathfrak{M}_{m,n}(z)\right) = \frac{\exp\left(W_m(z)\right)}{\exp(W_n(z))}$$

for all $m, n \in \mathbb{Z}$. By (1.20) (which is valid for every branch) we eliminate the exponential on the right-hand-side:

$$\exp\left(\mathfrak{M}_{m,n}(z)\right) = \frac{W_n(z)}{W_m(z)}. \tag{3.22}$$

From here, we can proceed in two ways. Either we substitute $\mathfrak{M}_{m,n}(z)+W_n(z)$ in place of $W_m(z)$, or we substitute $W_m(z) - \mathfrak{M}_{m,n}(z)$ in place of $W_n(z)$:

$$\exp\left(\mathfrak{M}_{m,n}(z)\right) = \frac{W_n(z)}{\mathfrak{M}_{m,n}(z) + W_n(z)},$$
$$\exp\left(\mathfrak{M}_{m,n}(z)\right) = \frac{W_m(z) - \mathfrak{M}_{m,n}(z)}{W_m(z)}.$$

Then these equations can be solved for $W_n(z)$, and $W_m(z)$, respectively:

$$W_n(z) = \frac{\mathfrak{M}_{m,n}(z)}{\exp\left(-\mathfrak{M}_{m,n}(z)\right) - 1},$$

and

$$W_m(z) = \frac{\mathfrak{M}_{m,n}(z)}{1 - \exp\left(\mathfrak{M}_{m,n}(z)\right)}.$$

Take the exponential of both sides of the individual equations, and use the fact that $W_n(z)e^{W_n(z)} = z$. We therefore get that if $x = \mathfrak{M}_{n,m}(z)$, this x indeed solves (3.21).

We still have to show that if x is a solution, then $x = \mathfrak{M}_{m,n}(z)$ for some m and n. Equation (3.21) is of the form $ue^u = z$, so $u = W_n(z)$ for some integer n. In other words,

$$u = \frac{x}{e^{-x} - 1} = W_n(z). \tag{3.23}$$

If we want to show that $x = W_m(z) - W_n(z)$, it is better to fix x in the form

$$x = v - W_n(z), \tag{3.24}$$

and try to show that $v = W_m(z)$. Multiply both sides of the last displayed equation by minus one, and take the exponential:

$$e^{-x} = e^{W_n(z)}e^{-v} = \frac{z}{W_n(z)}e^{-v}.$$

Substitute this and (3.24) into (3.23). It turns out that the resulting equation is solvable in terms of the W function:

$$v = W_m(z), \quad \text{for some } m.$$

This is what we wanted to prove.

3.5.2 An equation for $\mathfrak{M}_{0,-1}(z)$

There is another special equation which can be solved by the branch difference function, but now we only need the parameters $m = 0$, $n = -1$.

The equation

$$xe^{-x\coth x}\operatorname{csch} x = z \tag{3.25}$$

has a real solution for $0 < z \le 1/e$:

$$x = \pm\frac{1}{2}\mathfrak{M}_{0,-1}(-z). \tag{3.26}$$

In the equation coth is the *hyperbolic cotangent*:

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1},$$

and csch is the *hyperbolic cosecant*:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}.$$

The proof is as follows. An inspection of Figure 29 shows that the function on the left-hand side of (3.25) is even in x, thus it suffices to consider only the set $x \ge 0$. Moreover, the function takes values from the interval $]0, -1/e]$; therefore, we see that the restriction on z is necessary. The maximum is taken for $x = 0$, where the value of the left-hand side is $1/e$.

If z is from the interval $]0, 1/e]$, the range of $\mathfrak{M}_{0,-1}(-z) = W_0(-z) - W_{-1}(z)$ is $[0, \infty[$ (this follows from (1.55)-(1.56)). This is in agreement with our restriction that $x \ge 0$.

Since the left-hand side is strictly decreasing over the positive axis, it is enough to show that $\frac{1}{2}\mathfrak{M}_{0,-1}(-z)$ is indeed a solution (over the positive reals). Substituting (3.26) into $x\operatorname{csch} x$, we get that

$$x\operatorname{csch} x = \frac{\mathfrak{M}_{0,-1}(-z)}{\exp(\mathfrak{M}_{0,-1}(-z)/2) - \exp(-\mathfrak{M}_{0,-1}(-z/2))}.$$

By (3.22), the right-hand-side is rewritten as

$$\frac{\mathfrak{M}_{0,-1}(-z)}{\exp(\mathfrak{M}_{0,-1}(-z)/2) - \exp(-\mathfrak{M}_{0,-1}(-z/2))} = -W_0(-z)\sqrt{\frac{W_{-1}(-z)}{W_0(-z)}} =$$

$$= -W_0(-z)\sqrt{\frac{W_{-1}(-z)W_0(-z)}{W_0^2(-z)}} = \sqrt{W_{-1}(-z)W_0(-z)}.$$

(Care needs to be taken when manipulating the square roots.) We thus get that

$$x\operatorname{csch} x = \sqrt{W_{-1}(-z)W_0(-z)}.$$

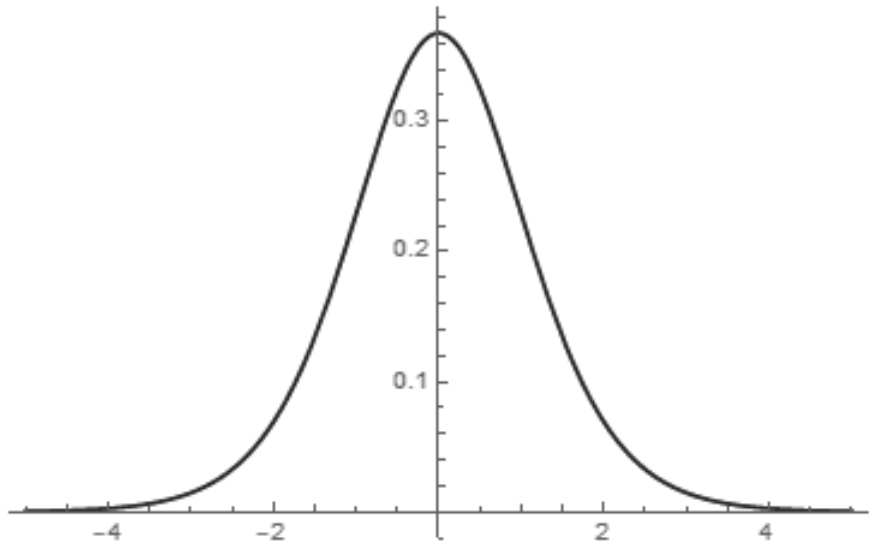


FIGURE 29
The plot of $xe^{-x\coth x}\operatorname{csch} x$ on the interval $[-5,5]$.

With the same approach, substituting (3.26) into $e^{-x\coth x}$:

$$e^{-x\coth x}=\sqrt{\frac{z^2}{W_0(-z)W_{-1}(-z)}}.$$

Multiplying what we have got for $x\operatorname{csch} x$ and for $e^{-x\coth x}$, the statement follows.

Finally, we remark that, upon substituting $x=iy$, and recalling the identities

$$\coth(ix)=-i\cot x,$$
$$\operatorname{csch}(ix)=-i\csc x=-\frac{i}{\sin z},$$

equation (3.25) can be put in another form:

$$\frac{y}{\sin y}e^{-y\cot y}=z, \tag{3.27}$$

and the solution is then

$$y=\pm\frac{i}{2}\mathfrak{M}_{0,-1}(-z).$$

The range of $\mathfrak{M}_{0,-1}(z)$

In [91] the range of the function

$$\mathfrak{M}_{0,-1} : \mathbb{C} \to \mathbb{C}$$

was determined. This function maps $\mathbb{C}$ onto the region of $\mathbb{C}$ surrounded by the positive real and imaginary axes, and by the two curves

$$C_1 = \{W_0(x) - W_{-2}(x) : x \in [-1/e, 0[\},$$
$$C_2 = \{W_{-1}(x) - W_{-2}(x) : x \in]-\infty, -1/e[\}.$$

See Figure 30 where this region is depicted.

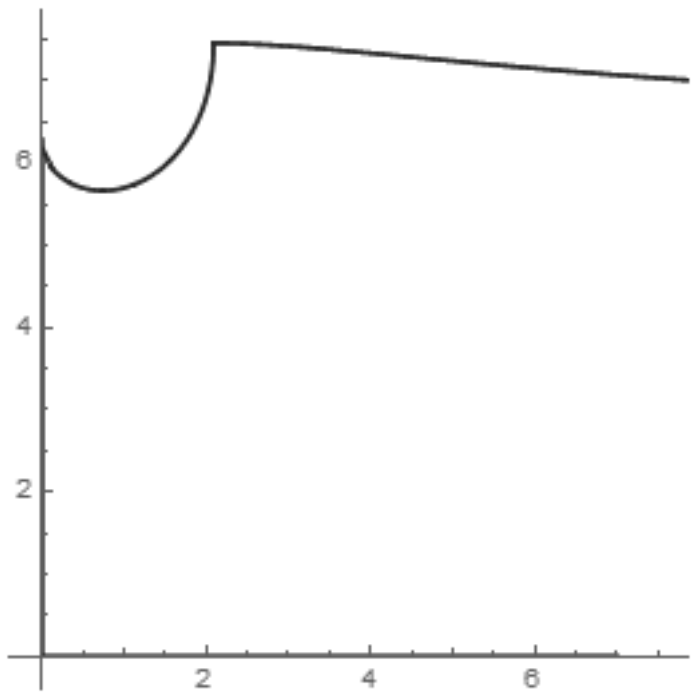


FIGURE 30
The range of the branch difference function $\mathfrak{M}_{0,-1}$. The vertical line segment goes up to $2\pi i$. The wall is the curve C_2 and the upmost curve reaching out to plus infinity is C_1.

Further notes

1. **Computer algebra.** We saw in this chapter that it is often suggested to unlearn our high-school routine when doing calculations with complex numbers, because usual identities might fail. At the dawn of computer programming, even professional programmers wrote erroneous codes. N. Higham [81] notes that it took five papers and three years to get the complex logarithm in the Algol 60 language right. H. Aslaksen [10] gave tests for users of mathematical software to check the capabilities of their systems. The introduction of the unwinding number and agreement on branch cuts of the usual inverse functions ($\sin^{-1}$, $\sinh^{-1}$, etc.) highly uniformized and simplified the programming; see [49, 99] for some historical insight, and [25, 40, 87, 88, 99, 146] for practical suggestions for computer algebra systems.

   About the implementation of the Lambert function in Maple, see [89].

2. **A physical application of equation** (3.27)**.** Although (3.27) seems to be rather artificial, it *does* have applications: it occurs in the study of dispersion equations for damped plasma oscillations. See [5] for details, especially around equation (15) there.

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

4

Numerical approximations

We have encountered a large number of nice properties of the Lambert W function: we know that it solves equations with exponentials and logarithms, it has nice integral representations as (1.59), and we know interesting closed-form evaluations at some arguments, like (1.27). It is sometimes also good to know how a function behaves *approximately* for large arguments, or for arguments around critical points like branch points. In this chapter we address these questions: we first derive a nice infinite series representation for W, then study its Padé and continued fraction approximations.

4.1 A concise series representation of W_k

The most common way of calculating numerical – real or complex – solutions of equations of the form $f(w) = z$ is to apply some iterative root-finding method, like the Newton method, Halley method or some more specialized approach, depending on the properties of the function f. In our case, $f(w) = we^w$, which is not a complicated function, and empirical examination tells us [39] that the Halley method is perfect to numerically determine the values of W. One needs a good *initial guess* from where to start the iteration. The goodness of the initial guess and the rapidity of convergence depend on the characteristics of the computer architecture we use, and the built-in numerical algorithms of the software we choose. This means that there is room for experimentation. During the writing of the *Maple* software code for the W function, such experimentation was explicitly admitted [39, p. 355], but not all the results of this experimentation were disclosed in [39]. There exists general, device independent analysis for the calculation of W [94].

We first deduce a very nice series representation for W on all the branches, which is interesting in itself and the first two terms of this series give good approximation of $W_k(z)$ for every k and large z.

4.1.1 A first approximation of W

We now make our first step toward the full description of the asymptotic behavior of the W function.

DOI: 10.1201/9781003168102-4

W is the inverse of we^w. The inverse of the simpler e^w is just the logarithm, and the logarithm function grows, well, logarithmically. How does we^w grow then? Since, for large w, we^w grows more rapidly than the exponential, we expect its inverse to grow more *slowly* than the logarithm. Therefore, we write

$$W(z) = \log z + u(z), \tag{4.1}$$

and try to find an approximate description of the "sublogarithmic" term $u(z)$. In the below approach, we follow de Bruijn [47, p. 25-28], and the paper [39] (de Bruijn was only interested in the positive real solution of the Lambert equation $xe^x = y$, [39] generalizes his argument to the complex setting). Putting (4.1) into the defining equation $z = W(z)e^{W(z)}$:

$$z = (\log z + u(z))e^{\log z + u(z)} = (\log z + u(z))ze^{u(z)}.$$

That is,

$$\left(1 + \frac{u(z)}{\log z}\right) e^{u(z)} = \frac{1}{\log z}.$$

According to our assumption, $|u(z)| \ll |\log z|$, and thus

$$\left(1 + \frac{u(z)}{\log z}\right) e^{u(z)} \approx e^{u(z)}.$$

This means that

$$u(z) \approx \log \frac{1}{\log z}. \tag{4.2}$$

We try to make our deduction valid on the complex plane, so we must remember that the logarithms in this approximate form of $u(z)$ might denote two *different branches*. However, if we want $u(z)$ to be small compared to $\log(z)$, the outer logarithm should not have a large branch index m; otherwise, for moderate z, $\log_m \frac{1}{\log z}$ would have a large imaginary part. Therefore, we fix m to be zero, so the outer logarithm is taken to be the *principal logarithm*. As it is customary in the literature, the logarithm with not-yet-chosen branch index is denoted by Log. We therefore rewrite (4.2) as

$$u(z) \approx \log \frac{1}{\operatorname{Log} z}.$$

One observation will be handy here. It is not true in general, that $\log(1/z) = -\log z$, but only when $\mathcal{K}(-\log z) = 0$ (see (3.9)). Thus,

$$\log \frac{1}{\operatorname{Log} z} = -\log \operatorname{Log} z \quad \Leftrightarrow \quad \mathcal{K}(-\log \operatorname{Log} z) = 0.$$

When does this condition on the unwinding number satisfy? According to the (3.6) definition of $\mathcal{K}$, it must hold true that

$$-\pi < \Im(-\log \operatorname{Log} z) \leq \pi,$$

but the imaginary part of $-\log \operatorname{Log} z$ is $-\arg(\operatorname{Log} z)$. This falls on the interval $]-\pi, \pi]$ only when $\arg(\operatorname{Log} z) \neq \pi$. The argument of $\operatorname{Log} z$ can be equal to π only if $\operatorname{Log} = \log$, the principal branch, and we also must have that $\log z$ is a negative real number. This only happens if $0 < z < 1$. Therefore, we get the result that

$$\log \frac{1}{\operatorname{Log} z} = -\log \operatorname{Log} z \quad (z \in \mathbb{C} \backslash]0, 1[).$$

Hence, we continue the approximation (4.1) as

$$W(z) = \operatorname{Log} z - \log \operatorname{Log} z + v(z). \tag{4.3}$$

Thus, we split the function u of (4.1) into a sublogarithmic part, plus a new part v. Some complex function theoretic machinery will help to find an integral representation for v, and this will lead to the full asymptotic expansion of the W function.

4.1.2 The integral representation of $v(z)$

Upon substituting (4.3) into $W(z)e^{W(z)} = z$, we get

$$(\operatorname{Log} z - \log \operatorname{Log} z + v)\frac{z}{\operatorname{Log} z}e^{v} = z.$$

Let us introduce the below abbreviations (following de Bruijn):

$$\frac{1}{\operatorname{Log} z} = \sigma, \quad \frac{\log \operatorname{Log} z}{\operatorname{Log} z} = \tau. \tag{4.4}$$

With these, the last displayed equation becomes

$$e^{-v} - 1 + \tau - v\sigma = 0. \tag{4.5}$$

Here we suppose that neither $\operatorname{Log} z$ nor z is zero. Moreover, we consider τ and σ as independent variables, although they are clearly not independent. This will not cause any problem in the deduction.

Our program is as follows. We fix a branch for Log, and thus we implicitly fix σ and τ. We will show that for these σ and τ, (4.5) has a unique solution v. With the help of some complex analysis, we will find a series expansion for v, thus getting a series representation for $W(z)$ by (4.3).

Let a branch be fixed for Log. Then there is a positive real number, such that

$$|\sigma| < a, \quad |\tau| < a.$$

Notice that if z is large enough, a is a small number. This observation will be of importance soon.

Next, consider the set

$$\{|e^{-\zeta} - 1| : |\zeta| = \pi\},$$

and let δ be the lower bound of this set of positive real numbers. (δ, in fact, equals $1 - e^{-\pi} \approx 0.9568$.) The function $e^{-\zeta} - 1$ has a unique root inside the disk $|\zeta| < \pi$, and this root is $\zeta = 0$.

As we said earlier, for an appropriate choice of z, the bound a above can be chosen to be small. Let us choose a z such that

$$a = \frac{\delta}{2(\pi + 1)}.$$

Then we have, from the inequality $|x - y| < |x| + |y|$, that

$$|\sigma\zeta - \tau| < |\sigma\zeta| + |\tau| = |\sigma||\zeta| + |\tau| <$$

$$\pi\frac{\delta}{2(\pi + 1)} + \frac{\delta}{2(\pi + 1)} = (\pi + 1)\frac{\delta}{2(\pi + 1)} = \frac{\delta}{2}.$$

It comes that

$$|\sigma\zeta - \tau| < \frac{\delta}{2} < |e^{-\zeta} - 1|$$

on the circle $|\zeta| = \pi$. (The radius π does not play a special role here, any number smaller than 2π would equally be perfect. We follow the choice of de Bruijn.)

We now recall Rouché's theorem [35, p. 125]: if f and g some analytic functions inside some bounded region R, and $|g(z)| < |f(z)|$ on the boundary of R, then f and $f + g$ have the same number of roots inside R.

Applying this theorem to the functions $g(\zeta) = \sigma\zeta - \tau$, and $f(\zeta) = e^{-\zeta} - 1$ on the region $R = \{\zeta : |\zeta| < \pi\}$, we get that $g(\zeta) + f(\zeta) = e^{-\zeta} - 1 + \tau - \zeta\sigma$ has exactly one root, say v, on the open disk R. This is what we wanted to prove for equation (4.5).

In the next step we give an integral representation for v. Since v is the unique simple zero of the analytic function $h(\zeta) = e^{-\zeta} - 1 + \tau - \zeta\sigma$ in R (σ and τ fixed), the function $\frac{h'(\zeta)}{h(\zeta)}(\zeta - v)$ has neither zeros nor poles in R. In particular, by Cauchy's theorem,

$$\int_{\partial R} \frac{h'(\zeta)}{h(\zeta)}(\zeta - v)\mathrm{d}\zeta = 0.$$

Or, equivalently,

$$v \int_{\partial R} \frac{h'(\zeta)}{h(\zeta)}\mathrm{d}\zeta = \int_{\partial R} \frac{h'(\zeta)}{h(\zeta)}\zeta\mathrm{d}\zeta. \tag{4.6}$$

Here ∂R stands for the boundary circle of the region R, taken counterclockwise.

Recall the argument principle [35, p. 123]: the integral

$$\frac{1}{2\pi i} \int_{\partial R} \frac{h'(\zeta)}{h(\zeta)}\mathrm{d}\zeta = Z - P,$$

where Z is the zeros, and P is the number of poles of the function h in R. In our case $Z = 1$, and $P = 0$, so (4.6) is rewritten as

$$v = \frac{1}{2\pi i}\int_{\partial R}\frac{h'(\zeta)}{h(\zeta)}\zeta \mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\partial R}\frac{-e^{-\zeta}-\sigma}{e^{-\zeta}-1+\tau-\zeta\sigma}\zeta \mathrm{d}\zeta. \tag{4.7}$$

We have got an important statement: for any branch choice of Log and large enough z, there is a unique $v \in R$ such that $W(z)e^{W(z)} = z$ with (4.3); and this v is expressed by the integral (4.7). This statement establishes the existence of the infinite number of roots $W_k(z)$, something we already know from Chapter 2.

Remember that

$$|e^{-\zeta}-1| > |\tau-\sigma\zeta|, \tag{4.8}$$

so the integral on the right-hand side of (4.7) has no singularity on the circle ∂R. The integrand is rewritten as

$$\frac{-e^{-\zeta}-\sigma}{e^{-\zeta}-1+\tau-\zeta\sigma}\zeta = \frac{1}{e^{-\zeta}-1}\frac{-e^{-\zeta}-\sigma}{1+\frac{\tau-\zeta\sigma}{e^{-\zeta}-1}}\zeta.$$

Because of (4.8), the fraction $\frac{\tau-\zeta\sigma}{e^{-\zeta}-1}$ can be expanded as a double series with terms of the form

$$\frac{\zeta^k\sigma^k\tau^m}{(e^{-\zeta}-1)^{k+m+1}}A_{k,m},$$

with $A_{k,m}$ some constants. Upon integrating such a series term-by-term (the term-by-term integration can be done, because the absolute convergence is assured by (4.8)), the variable ζ is integrated out, and we get a double series purely in terms of σ and τ:

$$v = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}c_{k,m}\sigma^k\tau^m. \tag{4.9}$$

4.1.3 The $c_{k,m}$ constants

In order to find the constants $c_{k,m}$, we use the fact that v is the solution of (4.5), and write

$$f(v) = 1 - e^{-v} + \sigma v = \tau.$$

We invert this equation by the Lagrange inversion theorem (see (1.51) in Section 1.4). Putting $x_0 = 0$, $y = \tau$, and $x = v$ in (1.51):

$$v = \sum_{m=1}^{\infty} g_m\frac{\tau^m}{m!}, \tag{4.10}$$

where

$$g_m = \frac{\partial^{m-1}}{\partial v^{m-1}}\left(\frac{v}{1-e^{-v}+\sigma v}\right)^m\Bigg|_{v=0} = \frac{\partial^{m-1}}{\partial v^{m-1}}\frac{1}{\left(\frac{1-e^{-v}}{v}+\sigma\right)^m}\Bigg|_{v=0}. \tag{4.11}$$

In the next step, we write g_m as a series of σ. We make use of the series expansion

$$\frac{1}{(h+\sigma)^m} = \sum_{k=0}^{\infty} h^{-m-k} \binom{-m}{k} \sigma^k.$$

Applying this to (4.11) with $h = \frac{1-e^{-v}}{v}$:

$$\frac{1}{\frac{1-e^{-v}}{v}+\sigma} = \sum_{k=0}^{\infty} \left(\frac{v}{1-e^{-v}}\right)^{m+k} \binom{-m}{k} \sigma^k.$$

Then, for g_m we now have that

$$g_m = \frac{\partial^{m-1}}{\partial v^{m-1}} \sum_{k=0}^{\infty} \left(\frac{v}{1-e^{-v}}\right)^{m+k} \binom{-m}{k} \sigma^k \Bigg|_{v=0} =$$

$$\sum_{k=0}^{\infty} \binom{-m}{k} \sigma^k \frac{\partial^{m-1}}{\partial v^{m-1}} \left(\frac{v}{1-e^{-v}}\right)^{m+k} \Bigg|_{v=0}.$$

Notice that this is in agreement with (4.9): substituting our last expression of g_m into (4.10), the series of v contains a double sum, involving powers of σ and τ and some constants $c_{k,m}$. We still need to find

$$c_{k,m} = \binom{-m}{k} \frac{\partial^{m-1}}{\partial v^{m-1}} \left(\frac{v}{1-e^{-v}}\right)^{m+k} \Bigg|_{v=0}. \tag{4.12}$$

Our statement is that

$$c_{k,m} = (-1)^k \begin{bmatrix} k+m \\ k+1 \end{bmatrix}. \tag{4.13}$$

Here $\begin{bmatrix} k+m \\ k+1 \end{bmatrix}$ is an *unsigned Stirling number of the first kind*.

The function $\left(\frac{v}{1-e^{-v}}\right)^{m+k}$ has a simple series expansion, once we know the *Stirling polynomials* $S_k(x)$. These are defined as the Taylor coefficients of $\left(\frac{t}{1-e^{-t}}\right)^{x+1}$:

$$\left(\frac{t}{1-e^{-t}}\right)^{x+1} = \sum_{k=0}^{\infty} S_k(x) \frac{t^k}{k!}.$$

It then comes that

$$\left(\frac{v}{1-e^{-v}}\right)^{m+k} = \sum_{k=0}^{\infty} S_k(m+k-1) \frac{v^k}{k!}.$$

Therefore,

$$\frac{\partial^{m-1}}{\partial v^{m-1}} \left(\frac{v}{1-e^{-v}}\right)^{m+k} \Bigg|_{v=0} = S_{m-1}(m+k-1),$$

whence (4.12) becomes

$$c_{k,m} = \binom{-m}{k} S_{m-1}(m+k-1). \tag{4.14}$$

It is known that [149, Section 4.4.8]

$$S_k(m) = \frac{(-1)^{k+m-1}}{\binom{m}{k}} \begin{bmatrix} m+1 \\ m+1-k \end{bmatrix} \quad (m \in \mathbb{Z}, m \geq k).$$

By the application of this expression, (4.14) becomes

$$c_{k,m} = \frac{\binom{-m}{k}}{\binom{m+k-1}{m-1}} \begin{bmatrix} k+m \\ k+1 \end{bmatrix}.$$

Since

$$\binom{-m}{k} = (-1)^k \binom{m+k-1}{k},$$

we see that (4.13) is indeed valid. We get our new expression for v from (4.10):

$$v = \sum_{m=1}^{\infty} \frac{\tau^m}{m!} \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} k+m \\ k+1 \end{bmatrix} \sigma^k. \tag{4.15}$$

4.1.4 The asymptotic series representation for W_k

Our finding (4.15), together with (4.3), results in the wanted asymptotic series development of $W(z)$:

$$W(z) = \operatorname{Log} z - \log \operatorname{Log}(z) + \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{m!} \begin{bmatrix} k+m \\ k+1 \end{bmatrix} \frac{1}{\operatorname{Log}^k z} \left(\frac{\log \operatorname{Log} z}{\operatorname{Log} z} \right)^m.$$

Only one question is left open: which branch of Log must be chosen to represent a given branch of W? The imaginary part of the dominant term, $\operatorname{Log} z$, is $\arg z + 2k\pi$. A quick check of the images of the branches of the W function on Figure 13 shows that this is compatible with the branch index k of W. The approximation for a given choice of branch of Log must represent a given branch of W, so if it holds asymptotically, it must hold everywhere. Thus k cannot independently vary on the left- and right-hand sides. We therefore get our concise asymptotic series expansion for the W function:

$$\boxed{W_k(z) = \log_k z - \log \log_k(z) + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{m!} \begin{bmatrix} n+m \\ n+1 \end{bmatrix} \frac{(\log \log_k z)^m}{\log_k^{n+m} z}} \tag{4.16}$$

(We changed the summation index to avoid clashes with the branch index k.) Here $\log_k$ is the k-th branch of the logarithm, see (2.3).

The analysis of the convergence domain is not straightforward at all. This task was done by G. A. Kalugin and D. J. Jeffrey [100]. They showed that if $z \in \mathbb{R}$, the above series is convergent for $z > e$. For complex z, the series converges on the domain given by the implicit inequality

$$\Re W_m\left(-\frac{\log z}{e}\right) > 1.$$

Here

$$m = \begin{cases} -1, & -\pi < \arg z \le 0 \\ 1, & 0 < \arg z \le \pi \end{cases}.$$

For further reference, it is useful to specialize (4.16) to the principal branch, keeping just the first three terms:

$$\boxed{W_0(z) = \log z - \log\log(z) + \frac{\log\log z}{\log z} + O\left(\left(\frac{\log\log z}{\log z}\right)^2\right).} \tag{4.17}$$

4.1.5 The analysis of the remainder term in the asymptotic series representation for W_k

It is good to know how much we lose if we truncate the series (4.16) somewhere after finitely many terms. First, we analyze the magnitude of the coefficients $d_{k,m} = \frac{(-1)^k}{m!}\left[{k+m \atop k+1}\right]$. For all $k, m \ge 0$ it holds true that

$$|d_{k,m}| \le 4^{k+m}. \tag{4.18}$$

To prove this, we need to establish an estimation for the Stirling numbers[1]

$$\left[{n \atop k}\right] \le \frac{2^n n!}{k!}. \tag{4.19}$$

Recall that the Stirling numbers satisfy the recurrence

$$\left[{n+1 \atop k}\right] = n\left[{n \atop k}\right] + \left[{n \atop k-1}\right].$$

The inequality holds for $\left[{2 \atop 1}\right] = \left[{2 \atop 2}\right] = 1$. Next, we apply the recurrence to write

$$\left[{n+1 \atop k}\right] \le n\frac{2^n n!}{k!} + \frac{2^n n!}{(k-1)!} = \frac{2^n n!}{(k-1)!}\left(\frac{n}{k} + 1\right) =$$

$$\frac{2^n n!}{(k-1)!}\frac{n+k}{k} \le \frac{2^n n!}{(k-1)!}\frac{2(n+1)}{k} = \frac{2^{n+1}(n+1)!}{k!}.$$

[1] Better approximations are known for the Stirling numbers of the first kind, see [184].

Apart of the induction hypothesis, we used the fact that $k \leq n$ for the non-zero Stirling numbers.

To prove (4.18), we use (4.19) as follows:

$$|d_{k,m}| = \frac{1}{m!}\begin{bmatrix} k+m \\ k+1 \end{bmatrix} \leq \frac{2^{k+m}(k+m)!}{m!(k+1)!} \leq$$

$$\frac{2^{k+m}(k+m)!}{m!k!} = 2^{k+m}\binom{k+m}{m} \leq 2^{k+m}2^{k+m} = 4^{k+m}.$$

We are now ready to estimate the remainder term in the truncation

$$W_k(z) = \log_k z - \log\log_k(z) + \sum_{m=1}^{M-1}\sum_{n=0}^{N-1}\frac{(-1)^n}{m!}\begin{bmatrix} n+m \\ n+1 \end{bmatrix}\frac{(\log\log_k z)^m}{\log_k^{n+m} z} + \varepsilon_{M,N}(z).$$

We prove that if $|\sigma| < 1/4$, $|\tau| < 1/4$, and in the case when $k = 0$, we require $|z| > 1$, then

$$|\varepsilon_{M,N}(z)| \leq \frac{4|\tau|(4|\sigma|)^N + (4|\tau|)^M}{(1-4|\sigma|)(1-4|\tau|)}. \tag{4.20}$$

(Remember, that σ and τ are defined as in (4.4).) Under the conditions for the magnitude of σ and τ, and by recalling (4.18),

$$|\varepsilon_{M,N}(z)| \leq \sum_{m=M}^{\infty}\sum_{n=0}^{\infty}(4|\sigma|)^n(4|\tau|)^m + \sum_{m=1}^{\infty}\sum_{n=N}^{\infty}(4|\sigma|)^n(4|\tau|)^m =$$

$$\frac{1}{1-4|\sigma|}\frac{4^M|\tau|^M}{1-4|\tau|} + \frac{4|\tau|}{1-4|\tau|}\frac{4^N|\sigma|^N}{1-4|\sigma|}.$$

From here, (4.20) follows.

The analysis of this subsection was done by Johansson [94].

4.2 Series expansions and Padé approximants around $-1/e$

The understanding of the behavior of an inverse function around its branch points is important for many practical reasons. In this section we develop approximations for W around the branch point $z = -1/e$.

4.2.1 The behavior of W_{-1} and W_1 around $z = -1/e$

At the point $z = -1/e$, the value of W_{-1} is minus one. Thus, we write $W(z)$ around this point as

$$W(z) = -1 + p(z).$$

Then

$$z = W(z)e^{W(z)} = (-1 + p(z))e^{-1+p(z)} = \frac{1}{e}(-1 + p(z))e^{p(z)}.$$

Since $p(z)$ is small, we interpolate $e^{p(z)}$ with $1 + p(z)$. We then get that

$$z = \frac{1}{e}(-1 + p(z))(1 + p(z)).$$

This has two solutions:

$$p_-(z) = -\sqrt{ez + 1}, \quad p_+(z) = \sqrt{ez + 1}. \tag{4.21}$$

To find the expression for $W(z)$, we invert

$$ez = W(z)e^{1+W(z)} = (-1 + p(z))(1 + p(z)).$$

To eliminate square roots of two in the resulting formulas, instead of (4.21) we set

$$p_-(z) = -\sqrt{2(ez + 1)}, \quad p_+(z) = \sqrt{2(ez + 1)}. \tag{4.22}$$

The Lagrange inversion now gives that

$$W(z) = -1 + p - \frac{1}{3}p^2 + \frac{11}{72}p^3 + \cdots \tag{4.23}$$

around $z = -1/e$.

These coefficients can be given by a recurrence relation. Set

$$W(z) = \sum_{l=0}^{\infty} \mu_l p^l. \tag{4.24}$$

Then $\mu_0 = -1$, $\mu_1 = 1$, and

$$\mu_k = \frac{k-1}{k+1}\left(\frac{\mu_{k-2}}{2} + \frac{\alpha_{k-2}}{4}\right) - \frac{\alpha_k}{2} - \frac{\mu_{k-1}}{k+1}.$$

Here the sequence (α_k) is again given by the sequence (μ_k):

$$\alpha_k = \sum_{j=2}^{k-1} \mu_j \mu_{k+1-j}$$

with $\alpha_0 = 2$, and $\alpha_1 = -1$. This method of calculation was observed by D. Coppersmith in a private communication to R. M. Corless et al. [44]. The basis of the deduction is the valid expression

$$2p(z)W(z) = \left(\frac{p^2(z)}{2} - 1\right)\frac{d}{dp}(1 + W(z))^2.$$

Singularity analysis shows that [94]

$$|\mu_l| = O\left(\frac{1}{(\sqrt{2})^l}\right).$$

This shows that the series in (4.24) converges whenever $|p| < \sqrt{2}$. Another, not optimal but explicit estimation:

$$|\mu_l| < 2\left(\frac{4}{5}\right)^l.$$

We have shown that if $z \approx -1/e$, the W function is close to $-1+p-1/3p^2$, but we have not shown *which* expression of p in (4.22) should be used. The following cases are there [39]:

- if we want to calculate $W_0(z)$, then use $p_+(z)$ in (4.23) to get the initial point for the Halley method,
- if $\Im(z) < 0$, $W_1(z)$ should be calculated by using $p_-(z)$,
- if $\Im(z) > 0$, $W_{-1}(z)$ should be calculated by using $p_-(z)$.

4.2.2 The notion of Padé approximant

The *Padé approximant* is a rational function approximation to a given function, where the degrees of the numerator and denominator are fixed. The Padé approximant of a function $f(x)$ of order $[L/M]$ (in a given point x) is the rational fraction

$$[L/M]_f(x) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_L x^L}{1 + b_1 x + b_2 x^2 + \cdots + b_M x^M}.$$

It can be proven that if the constant term in the denominator is one (as above), then the Padé approximant of any given order is *unique*.

The coefficients are determined from the requirement that $f(x)$ and its approximant $[L/M]_f(x)$ agree to the highest possible order in a given point x. This requirement is satisfied when

$$f(x) = [L/M]_f(x), \quad f'(x) = [L/M]'_f(x), \ldots,$$

$$f^{(L+M)}(x) = [L/M]_f^{(L+M)}(x).$$

In other words, we have that

$$f(x) - [L/M]_f(x) = O(x^{L+M+1}). \tag{4.25}$$

If these are satisfied, one can calculate the coefficients in $[L/M]_f(x)$ from the Taylor coefficients of $f(x)$ around a given point (here taken to be the origin). Up to $L = M = 1$, the Padé approximants for a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

are determined easily:

$$\begin{aligned} [1/0]_f(x) &= c_0 + c_1 x, \\ [0/1]_f(x) &= \frac{c_0}{1 - \frac{c_1}{c_0} x}, \\ [1/1]_f(x) &= \frac{c_0 + \frac{c_1^2 - c_0 c_2}{c_1} x}{1 - \frac{c_2}{c_1} x}. \end{aligned} \tag{4.26}$$

The arc tangent function

We show in an example why the Padé approximants – even of low degree – are often better than the Taylor series approximation.

We take $f(x) = \arctan x$. The Taylor expansion of this function is

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + O(x^7).$$

Depending whether we truncate after even or odd number of terms, the limit of the truncated series will be negative, or positive infinity, respectively. However, $\arctan x$ is bounded as $x \to \infty$:

$$\lim_{x \to \infty} \arctan x = \frac{\pi}{2} \approx 1.5708.$$

The [1/1] Padé approximant is

$$[1/1]_{\arctan}(x) = \frac{\frac{\pi}{4} - 1 + \left(\frac{\pi}{4} + 1\right) x}{1 + x},$$

and

$$\lim_{x \to \infty} [1/1]_{\arctan}(x) = 1 + \frac{\pi}{4} \approx 1.7854.$$

This is much closer to reality than plus or minus infinity! For the order [2/2] approximant we have the even better

$$\lim_{x \to \infty} [2/2]_{\arctan}(x) = \frac{6}{13} + \arctan 2 \approx 1.56869.$$

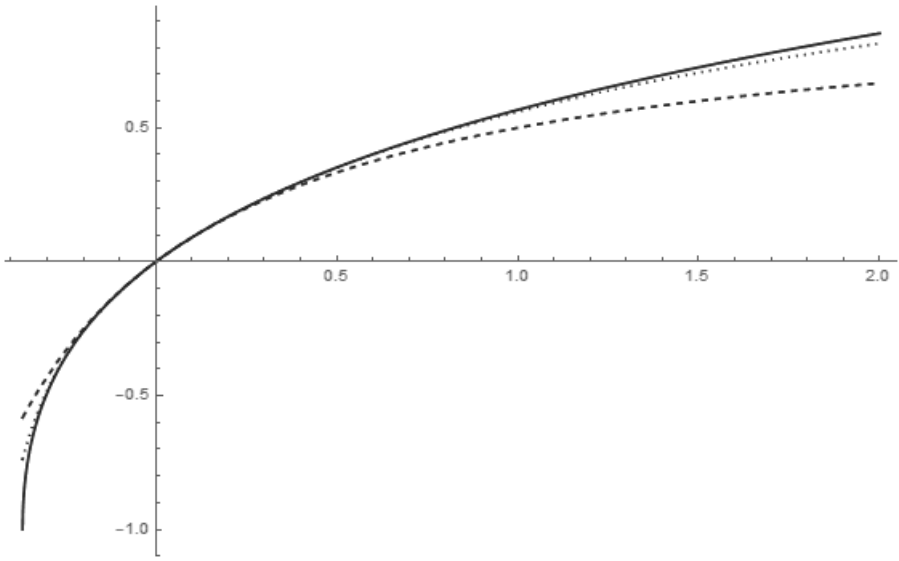


FIGURE 31
The plot of W_0 on the interval $[-1/e, 2]$ (solid), together with the plots of the $[1/1]_{W_0}$ and $[2/2]_{W_0}$ Padé approximants (dashed and dotted lines, respectively).

4.2.3 The Padé approximant of W_0 around zero

We give the first Padé approximants of W_0 around the origin:

$$
\begin{aligned}
[1/0]_{W_0}(0) &= x, \\
[1/1]_{W_0}(0) &= \frac{x}{1+x}, \\
[2/1]_{W_0}(0) &= \frac{x+\frac{1}{2}x^2}{1+\frac{3}{2}x}, \\
[2/2]_{W_0}(0) &= \frac{x+\frac{4}{3}x^2}{1+\frac{7}{3}x+\frac{5}{6}x^2}.
\end{aligned}
$$

Figure 31 contains plots which show the goodness of the $[1/1]_{W_0}$ and $[2/2]_{W_0}$ approximants.

4.3 Continued fraction representations for W_0

In this section we show another way of approximating (the principal branch of) the Lambert W function. We have already seen approximation which was similar to some extent: the Padé approximant. The Padé approximant is a a rational function approximation, so is the continued fraction expansion. We shall see what is the difference and similarity between these two in the following sections.

4.3.1 The notion of C-fractions

Let $(a_1, a_2, \ldots, a_n)$ be a finite sequence of non-zero real or complex numbers. A *regular, terminating or finite C-fraction* is the expression

$$\cfrac{a_1 z}{1 + \cfrac{a_2 z}{1 + \cfrac{a_3 z}{\ddots + a_n z}}}.$$

To shorten the typesetting of such fractions, we will use the following shorthand:

$$\cfrac{a_1 z}{1 + \cfrac{a_2 z}{1 + \cfrac{a_3 z}{\ddots + a_n z}}} = \operatorname*{K}_{m=1}^{n} \left(\frac{a_m z}{1} \right). \tag{4.27}$$

If $(a_1, a_2, \ldots)$ is an infinite sequence of non-zero numbers, we can consider the *non-terminating or infinite* continued fraction as a limit of the finite C-fractions:

$$\operatorname*{K}_{m=1}^{\infty} \left(\frac{a_m z}{1} \right) := \lim_{n \to \infty} \operatorname*{K}_{m=1}^{n} \left(\frac{a_m z}{1} \right),$$

if this limit exists. Therefore, a non-terminating regular C-fraction is an infinite continued fraction of the form

$$\cfrac{a_1 z}{1 + \cfrac{a_2 z}{1 + \cfrac{a_3 z}{1 + \ddots}}} =: \operatorname*{K}_{m=1}^{\infty} \left(\frac{a_m z}{1} \right).$$

(Non-regular C-fractions contain z's on powers different from one. *S-fractions* are C-fractions with strictly positive a_n.)

The a_m and other determining coefficients that we introduced later are called *elements*.

The n-th approximant of an infinite continued fraction attached to the sequence $(a_1, a_2, \dots)$ is the finite continued fraction (4.27). We say that a continued fraction *converges* in a point z, if the limit

$$\lim_{n\to\infty} \mathop{\mathrm{K}}_{m=1}^{n} \left(\frac{a_m z}{1}\right)$$

of the approximants exists.

In general, it is not easy to determine, whether a continued fraction converges. We give a theorem from 1904, due to Van Vleck [179], [97, Theorem 4.56] which is of particular importance for our purpose.

Let $\mathrm{K}_{m=1}^{\infty}(a_m z/1)$ be a regular C-fraction, such that

$$\lim_{m\to\infty} a_m = A \neq 0.$$

The continued fraction $\mathrm{K}_{m=1}^{\infty}(a_m z/1)$ converges to a function which is either meromorphic in

$$R_A = \left[z : \left| \arg\left(Az + \frac{1}{4}\right) \right| < \pi \right], \tag{4.28}$$

or it is identically infinity.

4.3.2 Correspondence between C-fractions and Taylor series

Our aim is to approximate the principal branch of the Lambert function by C-fractions, so we need to know how good the approximation is. One measure is the *order* to which the Taylor series of a function and the Taylor series of its continued fraction approximants agree. This is the same criteria that we used when we measured the goodness of the Padé approximant (see (4.25)). To make a precise definition, we need the notion of *correspondence*.

Let $f(z)$ be a function with a Taylor series around $z = 0$. Moreover, let

$$R_n(z) = \mathop{\mathrm{K}}_{m=1}^{n} \left(\frac{a_m z}{1}\right) \tag{4.29}$$

be the n-th approximant of a regular C-fraction. Let the Taylor series of the function $R_n(z)$ around $z = 0$ be $T_n(z)$. The *order of correspondence* of the approximant $R_n(z)$ is the exponent of the first non-vanishing power of z in the Taylor series of $f(z) - T_n(z)$.

In other words, if the n-th approximant $R_n(z)$ corresponds to $f(z)$ with order m, then the Taylor series of $R_n(z)$ and $f(z)$ agree term-by-term up to z^{m-1}.

The following theorem holds true [97, Corollary 5.3]. Every C-fraction

$$\mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m z}{1}\right)$$

corresponds to a uniquely determined Taylor series of the form

$$c_1 z + c_2 z^2 + c_3 z^3 + \cdots .$$

The order of correspondence of the nth approximant of (4.3.2) is $n+1$ (i.e., the first n terms agree in the Taylor expansions).

An example for the order of correspondence

To see an illustration of the above theorem, we fast-forward to the results of the next section, and take the third approximant of $W_0(x)$ from (4.37):

$$\cfrac{x}{1+\cfrac{x}{1+x/2}} = \frac{x(2+x)}{2+3x}.$$

The Taylor expansion of this rational fraction starts as

$$x - x^2 + \frac{3}{2}x^3 - \frac{9}{4}x^4 + \frac{27}{8}x^5 + O\left(x^6\right), \tag{4.30}$$

while we know very well from (1.52) that

$$W_0(x) = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \frac{125}{24}x^5 - \frac{54}{5}x^6 + \frac{16\,807}{720}x^7 + O\left(x^8\right). \tag{4.31}$$

And indeed, (4.30) and (4.31) agree up to the first three terms.

Going one approximant further, we have

$$\cfrac{x}{1+\cfrac{x}{1+\cfrac{x/2}{1+5x/6}}} = \frac{2x(3+4x)}{6+14x+5x^2}.$$

The first terms in the Taylor series of this function are

$$x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \frac{179}{36}x^5 + O\left(x^6\right),$$

and this agrees with (4.31) one more term.

4.3.3 The C-fraction expansion of $W_0(x)$ – via Hankel determinants

Now that we have the basic knowledge on the C-fractions, we can start to calculate the (a_n) sequence for the C-fraction representation of $W_0(x)$. We are going to do this in two ways. The first is important from a theoretical point of view, but hard computationally; the other method is easier to do on

a computer, due to its recursive nature. Let us meet these approaches one-by-one. Clearly, both must give us the same answer, as the C-fraction expansion is unique. We start with the first, which will be called the *Hankel determinant approach*.

The *Hankel determinants* of size 0 of a sequence (c_n) are $H_0^{(r)} = 1$, and of size k are

$$H_k^{(r)} = \begin{vmatrix} c_r & c_{r+1} & \cdots & c_{r+k-1} \\ c_{r+1} & c_{r+2} & \cdots & c_{r+k} \\ \vdots & \vdots & \vdots & \vdots \\ c_{r+k-1} & c_{r+k} & \cdots & c_{r+2k-2} \end{vmatrix} \quad (k, r \geq 1). \tag{4.32}$$

The following statement is valid [97, Theorem 7.2]. Let $f(z)$ be a function with Taylor series expansion

$$f(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots . \tag{4.33}$$

If

$$H_n^{(1)} \neq 0, \quad \text{and} \quad H_n^{(2)} \neq 0 \quad (n = 1, 2, \dots)$$

holds, and we define the sequence $(a_m)_{m\geq 1}$ such that $a_1 = H_1^{(1)}$, and

$$a_{2m} = -\frac{H_{m-1}^{(1)} H_m^{(2)}}{H_m^{(1)} H_{m-1}^{(2)}}, \quad a_{2m+1} = -\frac{H_{m+1}^{(1)} H_{m-1}^{(2)}}{H_m^{(1)} H_m^{(2)}}, \quad (m = 1, 2, \dots), \tag{4.34}$$

then the regular C-fraction

$$1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m z}{1} \right) \tag{4.35}$$

corresponds to (4.33).

By using this theorem, we can determine the wanted C-fraction for $W_0(x)$, we just set

$$c_0 = 0, \quad c_n = \frac{(-n)^{n-1}}{n!} \quad (n \geq 1), \tag{4.36}$$

and calculate the corresponding Hankel determinants. We get that $a_1 = 1$, and the numerical values of the first members of the even-indexed terms $(a_{2m})_{m\geq 1}$ are as follows:

$$1, \frac{5}{6}, \frac{133}{170}, \frac{13\,582\,711}{17\,940\,370}, \frac{10\,402\,118\,970\,990\,527}{14\,020\,256\,732\,380\,254},$$
$$\frac{83\,631\,044\,830\,029\,201\,279\,016\,528\,831}{114\,283\,391\,845\,598\,787\,583\,797\,734\,258},$$

while the first values of $(a_{2m+1})_{m\geq 1}$ are

$$\frac{1}{2}, \frac{17}{30}, \frac{1\,927}{3\,230}, \frac{92\,612\,482\,895}{150\,884\,743\,618}, \frac{59\,203\,666\,396\,198\,716\,260\,449}{94\,723\,672\,875\,246\,897\,516\,738},$$

$$\frac{1\,149\,522\,186\,344\,339\,904\,123\,210\,420\,373\,026\,673}{1\,816\,203\,966\,545\,767\,974\,156\,446\,391\,323\,945\,278}.$$

Since the Taylor series of $W_0(x)$ starts with x, we cancel the one in (4.35), and get that the C-fraction begins

$$W(x) = \cfrac{x}{1+\cfrac{x}{1+\cfrac{1/2x}{1+\cfrac{5/6x}{1+\cfrac{17/30x}{1+\ddots}}}}}. \tag{4.37}$$

4.3.4 The C-fraction expansion of $W_0(x)$ – via the *qd*-algorithm

The calculation of (a_m) via the (4.34) formulas is computationally expensive. H. Rutishauser gave a construction which is much simpler to perform [154, 155, 153]. This was dubbed "Quotienten-Differenzen-Algorithmus" by him (nowadays called *qd-algorithm*). The input is the sequence (c_n) of Taylor series coefficients from (4.33). Then, the *qd*-algorithm produces two internal sequences $(q_{m,n})$, and $(e_{m,n})$, as follows:

$$\begin{aligned} q_{1,n} &= \frac{c_{n+1}}{c_n}, \\ e_{0,k} &= 0, \\ e_{m,n} &= q_{m,n+1} - q_{m,n} + e_{m-1,n+1}, \\ q_{m,n} &= \frac{e_{m-1,n+1}}{e_{m-1,n}} q_{m-1,n+1}. \end{aligned}$$

From these, (a_m) is determined as follows:

$$a_{2m} = -q_m^{(1)} \quad (m = 1, 2, \dots); \tag{4.38}$$

$$a_{2m+1} = -e_m^{(1)} \quad (m = 1, 2, \dots). \tag{4.39}$$

The construction guarantees that (a_m) is the same as we got by the Hankel determinants in the previous subsection.

The below *Mathematica* code reproduces the *qd*-algorithm for the $W_0(x)$ function.

```
c[0] := 0
c[k_] := (-k)^(k - 1)/k!
q[1, n_] := c[n + 1]/c[n]
e[0, k_] := 0
e[m_, n_] := e[m, n] = q[m, n + 1] - q[m, n] +
```

```
                                    e[m - 1, n + 1]
q[m_, n_] := q[m, n] = e[m - 1, n + 1]/e[m - 1, n] *
                                    q[m - 1, n + 1]
```

According to (4.38)-(4.39), the output is produced by the code

```
Table[-q[m, 1], {m, 1, 8}]
Table[-e[m, 1], {m, 1, 8}]
```

4.3.5 Associated continued fractions

The *associated continued fraction* is a continued fraction of the form

$$\frac{a_1 z}{1 + b_1 z + \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{-a_m z^2}{1+b_m z} \right)}. \tag{4.40}$$

The associated continued fraction, as we see, involves two sequences. The payoff is that these continued fractions have more favorable properties than the C-fraction. Namely, the following theorem holds true [97, Theorem 7.13]. Every associated continued fraction (4.40) corresponds to a uniquely determined Taylor series of the form

$$1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots .$$

The order of correspondence of the nth approximant is $2n + 1$ (i.e., the first $2n$ terms agree in the Taylor expansions).

To illustrate the above statement, we take the approximant

$$\frac{x}{1 + x - \dfrac{x^2/2}{1 + 4x/3}} = \frac{2x(4x + 3)}{x(5x + 14) + 6}$$

from (4.41), and expand it around $x = 0$:

$$\frac{2x(4x + 3)}{x(5x + 14) + 6} = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \frac{179}{36}x^5 - \frac{1013}{108}x^6 + O(x^7).$$

This agrees with (4.31) up to four terms. Th next approximant is

$$\frac{x}{1 + x - \dfrac{x^2/2}{1 + 4x/3 - \dfrac{17x^2/36}{1 + 344x/255}}} = \frac{3x(x(451x + 912) + 340)}{x(x(665x + 3579) + 3756) + 1020}.$$

The Taylor series is now

$$x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \frac{125}{24}x^5 - \frac{54}{5}x^6 + \frac{2\,843\,701}{122\,400}x^7 + O(x^8),$$

and the first six coefficients agree.

Associated continued fractions via Hankel determinants

To express the coefficients (a_m) and (b_m), we need the following sequence of determinants. Let $h_0 = 0$, $h_1 = c_1$, and

$$h_k = \begin{vmatrix} c_1 & c_2 & \cdots & c_{k-1} & c_{k+1} \\ c_2 & c_3 & \cdots & a_k & c_{k+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_k & c_{k+1} & \cdots & c_{2k-2} & c_{2k} \end{vmatrix} \quad (k \geq 2).$$

(Here (c_i) is the sequence of coefficients in the Taylor series of the function we search the associated fraction for; in this case this sequence is (4.36).) Under the condition that the Hankel determinants (for their definition see p. 111)

$$H_n^{(1)} \neq 0 \quad (n = 1, 2, \dots),$$

we have the following expansion for the Lambert W function:

$$W(x) = \frac{x}{1 + x - \underset{m=2}{\overset{\infty}{\mathrm{K}}} \left(\frac{a_m x^2}{1 + b_m x} \right)},$$

where $a_i \neq 0$ $(i = 2, 3, \dots)$, and

$$a_m = \frac{H_m^{(1)} H_{m-2}^{(1)}}{\left(H_{m-1}^{(1)} \right)^2}, \quad b_m = \frac{h_{m-1}}{H_{m-1}^{(1)}} - \frac{h_m}{H_m^{(1)}}, \quad (m = 1, 2, \dots).$$

The numerical values of the first members of a_m starting from $m = 1$ are as follows:

$$1, \frac{1}{2}, \frac{17}{36}, \frac{13\,489}{28\,900}, \frac{16\,572\,760\,097}{35\,662\,811\,716}, \frac{142\,788\,351\,123\,209\,037\,281}{307\,919\,260\,288\,580\,642\,244},$$

$$\frac{25\,119\,358\,749\,156\,797\,880\,302\,687\,775\,459\,953}{54\,233\,971\,599\,633\,393\,837\,888\,667\,811\,199\,844},$$

while the first values of b_m $(m \geq 1)$ are

$$1, \frac{4}{3}, \frac{344}{255}, \frac{10\,864\,716}{8\,025\,955}, \frac{1\,123\,155\,307\,589\,936}{828\,449\,855\,279\,487},$$

$$\frac{1\,386\,150\,196\,983\,305\,151\,822\,858\,580}{1\,021\,631\,183\,985\,658\,216\,064\,479\,011},$$

$$\frac{52\,945\,666\,763\,455\,270\,214\,646\,517\,195\,662\,060\,951\,784\,968}{39\,004\,228\,094\,206\,727\,518\,004\,413\,560\,670\,847\,114\,045\,951}.$$

The associated continued fraction then begins

$$W(x) = \cfrac{x}{1 + x - \cfrac{1/2x^2}{1 + 4/3x - \cfrac{17/36x^2}{1 + 344/255x - \cfrac{13489/28900x^2}{\ddots}}}}. \tag{4.41}$$

Associated continued fractions via Gragg's algorithm

As with the C-fraction, it is possible to circumvent the Hankel determinants in the case of associated continued fractions, too: one can determine the sequences (a_m) and (b_m) *recursively*. The following algorithm is due to W. B. Gragg [73], but the ideas go back to Tchebycheff [172]; see also [97, p. 248].

Let (c_n) be given as in (4.36). We define five additional sequences.

$$\sigma_{-1} := 1, \quad \sigma_n := \sum_{j=0}^{n} b_{n,j} c_{2n+1-j},$$

$$\tau_{-1} := 0, \quad \tau_n := \frac{1}{\sigma_n} \sum_{j=0}^{n} b_{n,j} c_{2n+2-j},$$

$$k_n := \frac{\sigma_{n-1}}{\sigma_{n-2}} \quad (n \geq 1),$$

$$l_n := \tau_{n-2} - \tau_{n-1} \quad (n \geq 1),$$

$$b_{n,j} := \begin{cases} 0, & \text{if } n = j = 0; \\ 0, & \text{if } j = -1 \text{ or } j = n+1; \\ 1, & \text{if } j = 0; \\ b_{n-1,j} + l_n b_{n-1,j-1} - k_n b_{n-2,j-2}, & \text{if } 0 < j \leq n. \end{cases}$$

Then we have that

$$a_m = k_m \quad (m \geq 1), \tag{4.42}$$

$$b_m = l_m \quad (m \geq 1). \tag{4.43}$$

The *Mathematica* code realizes the above definitions[2].

```
c[n_] := (-n)^(n - 1)/n!
\[Sigma][-1] := 1
```

[2]In place of `\[Sigma]` and `\[Tau]` one should simply type the corresponding Greek letters directly in *Mathematica*. To get the σ character in the *Mathematica* editor, just

```
\[Tau][-1] := 0
\[Sigma][n_] := \[Sigma][n] =
                  Sum[b[n, j] c[2 n + 1 - j], {j, 0, n}]
\[Tau][n_] := \[Tau][n] =
  1/\[Sigma][n] Sum[b[n, j] c[2 n + 2 - j], {j, 0, n}]
k[n_] := k[n] = \[Sigma][n - 1]/\[Sigma][n - 2]
l[n_] := l[n] = \[Tau][n - 2] - \[Tau][n - 1]
b[0, 0] := 1
b[n_, j_] :=
 b[n, j] =
  Piecewise[{{0, j == -1 || j == n + 1}, {1,
     j == 0}, {b[n - 1, j] + l[n] b[n - 1, j - 1] -
      k[n] b[n - 2, j - 2], 0 < j <= n}}]
```

To list the (a_m) and (b_m) coefficients, one writes (remember (4.42)-(4.43))

```
Table[k[n], {n, 1, 8}]
Table[l[n], {n, 1, 8}]
```

4.3.6 The analysis of the relative errors of C- and associated fractions

After determining the elements of the C-fractions and associated fractions for W_0, we do some analysis regarding the relative errors of these approximations. We define

$$C(x) := \left|\frac{W(x) - CW(x)}{W(x)}\right|, \quad A(x) := \left|\frac{W(x) - AW(x)}{W(x)}\right|,$$

where

$$CW(x) = \frac{z}{1 + \underset{m=2}{\overset{16}{\mathrm{K}}} \left(\frac{a_m z}{1}\right)},$$

and

$$AW(x) = \frac{z}{1 + z - \underset{m=2}{\overset{16}{\mathrm{K}}} \left(\frac{a_m z^2}{1 + b_m z}\right)},$$

are the continued fractions on the right-hand side of (4.37) and (4.41), respectively, cut at the marked indices.

In the paper [37] we calculated $W(x)$ for a set of values from a wide range, then we determined $CW(x)$ and $AW(x)$. The results are in Tables 1-2.

press **<Esc>**, then type the word "sigma", then press **<Esc>** again.

x	$C(x)$	x	$C(x)$
$-1/e \approx -0.3679$	0.081	0	0.0
$-1/e + 0.01$	0.001867	0.1	0.0
$-1/e + 0.1$	7.8×10^{-9}	0.2	4.93×10^{-16}
-0.25	1×10^{-9}	0.5	2×10^{-11}
-0.1	1.2×10^{-16}	1	1.376×10^{-8}
-0.01	0.0	2	2.18×10^{-6}
		5	2.5×10^{-4}
		10	0.00288
		100	0.13

TABLE 1
Some values of $C(x)$.

x	$A(x)$	x	$A(x)$
$-1/e$	0.042	0	0.0
$-1/e + 0.01$	9.07×10^{-6}	0.1	0.0
$-1/e + 0.1$	1.39×10^{-16}	0.2	0.0
-0.25	0.0	0.5	1.578×10^{-16}
-0.1	1.24×10^{-16}	1	0
-0.01	0.0	2	3.59×10^{-12}
		5	4.8785×10^{-8}
		10	6.71×10^{-6}
		100	0.021

TABLE 2
Some values of $A(x)$.

m	a_m
2	1
10	0.741935
20	0.711317
50	0.692527
100	0.686123
200	0.682874
300	0.681782
400	0.681233
500	0.680902

TABLE 3
The even-indexed elements of the C-fraction (4.37).

m	a_m
1	1
11	0.625014
21	0.649959
51	0.666981
101	0.673113
201	0.676291
301	0.67737
401	0.677914
501	0.678243

TABLE 4
The odd-indexed elements of the C-fraction (4.37).

4.3.7 The asymptotics of the elements in the C-fractions and associated fractions

There are some interesting observations one can make about the asymptotic behavior of the elements of the two continued fraction representations that we have calculated.

The asymptotics of the elements in the C-fraction of W_0

We start with (4.37). Because of (4.38)-(4.39), we need to consider the even- and odd-indexed subsequences of (a_m) separately.

The behavior of the sequence (a_m) for even and odd indices seems to be as follows: a_{2m} slowly decreases, while a_{2m+1} slowly increases as m grows.

m	a_m
1	1
10	0.462441
20	0.461963
50	0.461839
100	0.461822
150	0.461819
200	0.461817
300	0.461817

TABLE 5
The a_m elements of (4.41).

It seems that they converge to the same constant c such that

$$a_{2m} \searrow c, \quad a_{2m+1} \nearrow c.$$

Calculating

$$\frac{a_{300} + a_{301}}{2} = 0.679576, \quad \frac{a_{400} + a_{401}}{2} = 0.6795735,$$

and

$$\frac{a_{500} + a_{501}}{2} = 0.6795725,$$

we might guess that the common limit c equals

$$c = \frac{e}{4} = 0.679570457\ldots. \tag{4.44}$$

We will scrutinize this limit in more details in the next subsection.

The asymptotics of the elements in the associated fraction of W_0

The a_m and b_m elements of (4.41) seem to converge somewhat faster.

It seems possible that

$$\lim_{m\to\infty} b_m = \frac{e}{4} \approx 1.3591409, \tag{4.45}$$

but for the limit of (a_m), we could not find a rational multiple of e that could fit well. When doing the calculations for the paper [37], the authors used the Online Encyclopedia of Integer Sequences (OEIS, [4]), as well as the Inverse Symbolic Calculator [2], but there was no matching between the approximative limit of (b_m) and numbers of the form re, where r would be a rational number.

m	b_m
1	1
10	1.35831
20	1.35893
50	1.35911
100	1.35913
150	1.35914
200	1.35914

TABLE 6
The b_m elements of (4.41).

4.3.8 The limit of the elements in the C-fraction expansion of W_0 and Van Vleck's theorem

Let us recall Van Vleck's theorem (p. 109) which says that if we have a regular C-fraction, such that for its elements

$$\lim_{m\to\infty} a_m = A \neq 0,$$

then the continued fraction converges to a function which is either meromorphic in the region (4.28), or it is identically infinity. We now try to connect this theorem with our conjecture (4.44). Let us suppose that $A = e/4$, put $z = s + it$, and check the region (4.28):

$$\left|\arg\left(\frac{e}{4}(s+it)+\frac{1}{4}\right)\right| < \pi.$$

The domain of the argument function is $]-\pi,\pi]$, so only those z complex numbers are excluded from the region for which

$$\arg\left(\frac{e}{4}(s+it)+\frac{1}{4}\right) = \pi.$$

But only negative integers have argument equal to π, so $t = 0$ must hold, and thus we have the simple equation

$$\frac{e}{4}s+\frac{1}{4} < 0,$$

and thus

$$s < -\frac{1}{e}.$$

And it is clear, a posteriori, that this must hold true: the continued fraction (4.37) represents W_0, but this function becomes complex if $x < -1/e$, while the continued fraction on the right-hand side of (4.37) is still real. Therefore, that continued fraction represents W_0 only for $x \geq -\frac{1}{e}$. This fact, therefore, reinforces our observation that (4.44) holds true.

Further notes

1. **Series representations for W.** There are many other approaches for giving asymptotic series representations for the Lambert function apart from (4.16). These series behave differently regarding their convergence. See [44, 90, 100, 101] for further details. To give an example, if one considers the variable $\zeta = \frac{1}{1+\sigma}$ and does the Lagrange inversion, the following result holds true [90]:

$$W(z) = L_1 - L_2 + \sum_{m\geq 1} \frac{\tau^m}{m!} \sum_{p=0}^{m-1} (-1)^{p+m-1} \zeta^{p+m} \left\{ \begin{matrix} p+m-1 \\ p \end{matrix} \right\}_{\geq 2}.$$

   This series is convergent in the real case if $x > 2$. For the full analysis of convergence on the complex plane, the reader may consult [100].

2. **Numerical calculation of W.** There are many authors looking for efficient and accurate numerical calculations of W, either by means of device dependent or independent methods. Some of these results concentrate on only one branch of W, but there are general approaches, too. Aside from the paper of Johansson [94] and Jeffrey [89], one can refer to the papers [24, 85] on approximations for the principal branch of W; [31, 64, 178] contain results for the two real branches W_{-1} and W_0. The book chapter [145, Section 6.11] gives numerical algorithm for the equation $x \log x = y$ without mentioning the Lambert function.

3. **Continued fractions.** We used the book [97] as a resource for Van Vleck's theorem, and for the correspondence theorem. More on the theory of continued fractions can be found in [181], too. For connection between continued fractions and special functions, one can refer to [46].

4. **Inequalities.** Inequalities for special functions are often important and lead to non-trivial and nice results. There are a few papers on inequalities with respect to the Lambert W function, see for example [7, 83, 169]. In [32] some inequalities for W_{-1} (see theorem 1 there) are applied in outage analysis of user cooperation.

5. **Open poblems.** Give a proof or a disproof for the conjecture (4.45). Find a closed-form expression for the limit of the sequence of elements (a_m) in the associated fraction expansion of W_0.

Part II

Generalized Lambert functions

5

Generalizations of the Lambert function

The Lambert function has surprisingly many applications in different areas of science. Still, there are problems that need functions that are more general than W. In this chapter we introduce some functions which are mainly studied in the literature because of their utility in applications.

5.1 The generalized Lambert function

5.1.1 The polynomial-exponential and rational-exponential type equations

Thinking on generalizations of the equation

$$ze^z = w,$$

we observe that the factor z on the left-hand side is a *polynomial*, albeit among the simplest ones. It would make sense to put a polynomial $p(z)$ in place of z. Thus we arrive at an equation

$$p(z)e^z = w.$$

Such an equation is named *polynomial-exponential type*. Another extension is when, in place of $p(z)$, we put a *rational function*. A rational function is a fraction of two polynomials. We therefore write the extension as

$$\frac{p(z)}{q(z)}e^z = w. \tag{5.1}$$

We refer to such an equations as *rational-exponential type*.

Notice that a Lambert-like equation with "additive polynomial part" would not result in a new problem. Take the equation

$$e^z + p(z) = w.$$

This can very easily be transformed into a polynomial-exponential type:

$$e^z + p(z) = w \Longrightarrow e^z = w - p(z) \Longrightarrow \frac{1}{q(z)}e^z = 1,$$

DOI: 10.1201/9781003168102-5

where

$$q(z) = w - p(z)$$

is a polynomial.

To reach the ultimate extension that we encounter in the literature, we put a polynomial $r(z)$ in place of the simple polynomial inside the exponential function.

$$\frac{p(z)}{q(z)} e^{r(z)} = w. \tag{5.2}$$

All of these extensions are not mere generalizations for their own sake; equations of the form (5.2) are used in applications as we shall see the in Part III of this book.

5.1.2 The definition of the generalized Lambert function

If we factor the polynomials p and q (which can always be done over the complex plane), the rational-exponential type equation (5.1) becomes

$$\frac{(z-t_1)(z-t_2)\cdots(z-t_n)}{(z-s_1)(z-s_2)\cdots(z-s_m)} e^z = w. \tag{5.3}$$

The solutions of this equation will be given by the branches of the *generalized Lambert function*. We introduce the notation

$$W\left(\begin{matrix} t_1 & t_2 & \dots & t_n \\ s_1 & s_2 & \dots & s_m \end{matrix}; w\right) \tag{5.4}$$

for the solutions of equation (5.3).

We call the t_i parameters *upper parameters*, the s_js are the *lower parameters*. If there is no p or q, we do not write indices in the corresponding places. For example, the solutions of

$$p(z)e^{r(z)} = w$$

are written in the form

$$W\left(\begin{matrix} t_1 & t_2 & \dots & t_n \\ & & & \end{matrix}; w\right)$$

provided that $p(z) = (z-t_1)(z-t_2)\cdots(z-t_n)$.

It will be useful to set the type of equation (5.3) and of function (5.4) to be (n, m). Thus, instead of writing the lengthier generalized Lambert W function, we will often write Lambert function or W function of type (n, m).

In this generality, not much is known about the (5.4) function. We first see the cases when n and m are one or absent. Then we turn to the equation of type $(1, 1)$ because this case is fully understood.

5.1.3 Simple cases

Let us first study the simplest versions of the generalized Lambert function. If there are no polynomials at all in (5.3), we get the equation $e^z = w$, so

$$\boxed{W\left(\,;w\right) = \log(w).}$$

This shows that the logarithm is a particular case of the generalized Lambert function.

Now let us see what happens if there is one upper parameter and there is no lower parameter. That is, we want to solve the equation

$$(z-t)e^z = w.$$

We make the substitution

$$z - t = y \quad (z = t + y),$$

and the equation becomes

$$ye^{t+y} = w, \quad \text{or, equivalently} \quad ye^y = we^{-t}.$$

This is solvable in terms of the classical Lambert function, thus we get that

$$y = W(we^{-t}),$$

and then

$$z = t + W(we^{-t}).$$

We therefore get that

$$\boxed{W\left(\begin{matrix}t\\;w\end{matrix}\right) = t + W(we^{-t}).}$$

In particular,

$$W\left(\begin{matrix}0\\;w\end{matrix}\right) = W(w).$$

It is similarly easy to see that the Lambert function of type $(0,1)$ is again the classical Lambert function:

$$\boxed{W\left(\,;w\right) = s - W\left(-\frac{e^s}{w}\right).}$$

The complication comes when we have more parameters. But before going to investigate these cases, we remark some simple transformation formulas for the generalized W function.

5.2 Transformations of rational-exponential type equations

It is clear that we can permute the upper parameters and lower parameters among themselves; no change is made in the equation:

$$\boxed{W\left(\begin{matrix}t_1,t_2,\ldots,t_n\\ s_1,s_2,\ldots,s_m\end{matrix};w\right)=W\left(\begin{matrix}t_{i_1},t_{i_2},\ldots,t_{i_n}\\ s_{j_1},s_{j_2},\ldots,s_{j_m}\end{matrix};w\right),}$$

where $i_1,\ldots,i_n$ and $j_1,\ldots,j_m$ are arbitrary permutations of the first n and m positive integers, respectively.

It is similarly obvious, that equal upper and lower parameters can be deleted:

$$\boxed{W\left(\begin{matrix}t,t_2,\ldots,t_n\\ t,s_2,\ldots,s_m\end{matrix};w\right)=W\left(\begin{matrix}t_2,\ldots,t_n\\ s_2,\ldots,s_m\end{matrix};w\right).}$$

Another, less trivial transformation is the following for $w\neq 0$:

$$\boxed{W\left(\begin{matrix}t_1,t_2,\ldots,t_n\\ s_1,s_2,\ldots,s_m\end{matrix};w\right)=-W\left(\begin{matrix}-s_1,-s_2,\ldots,-s_m\\ -t_1,-t_2,\ldots,-t_n\end{matrix};\frac{(-1)^{m+n}}{w}\right).} \tag{5.5}$$

The $n=m=1$ particular case is

$$W\left(\begin{matrix}t\\ s\end{matrix};w\right)=-W\left(\begin{matrix}-s\\ -t\end{matrix};\frac{1}{w}\right).$$

The proof of (5.5) is as follows. We start with (5.3), and take the reciprocal of both sides:

$$\frac{(z-s_1)(z-s_2)\cdots(z-s_m)}{(z-t_1)(z-t_2)\cdots(z-t_n)}e^{-z}=\frac{1}{w}.$$

We now make the substitution

$$-z=y, \tag{5.6}$$

so the last displayed equation transforms into

$$\frac{(-y-s_1)(-y-s_2)\cdots(-y-s_m)}{(-y-t_1)(-y-t_2)\cdots(-y-t_n)}e^{y}=\frac{1}{w},$$

which is equivalent to

$$\frac{(y+s_1)(y+s_2)\cdots(y+s_m)}{(y+t_1)(y+t_2)\cdots(y+t_n)}e^{y}=\frac{(-1)^{m+n}}{w}.$$

For this latter equation, the solution is

$$W\left(\begin{matrix}-s_1 & -s_2 & \ldots & -s_m\\ -t_1 & -t_2 & \ldots & -t_n\end{matrix};\frac{(-1)^{m+n}}{w}\right).$$

We still need to multiply this by minus one because of the substitution (5.6). After this step, one arrives at the right-hand-side of (5.5).

5.3 The $(1,1)$-type function and the r-Lambert function

The first case when the generalized Lambert function gives rise to a new function is when we have at least one upper and at least one lower parameter. We now study the $(1,1)$-type function $W\left(\begin{smallmatrix} t \\ s \end{smallmatrix}; w\right)$, and point out that we can incorporate t and s into one parameter which will be denoted by r. The arising function will be called the r-Lambert function.

Let us do the substitution

$$y = z - t$$

in the equation

$$\frac{z-t}{z-s} e^z = w, \tag{5.7}$$

for which the solution, by definition, is

$$z = W\left(\begin{matrix} t \\ s \end{matrix}; w\right).$$

We get that

$$\frac{y}{y+t-s} e^{y+t} = w,$$

or

$$\frac{y}{y-(s-t)} e^y = e^{-t} w.$$

Multiplying by $y-(s-t)$, a simple rearrangement gives the equation

$$ye^y - e^{-t} wy = e^{-t} w(t-s).$$

Introducing the variable

$$r = -e^{-t} w,$$

it comes that the last equation is the same as

$$ye^y + ry = r(s-t).$$

Set

$$u = r(s-t),$$

we finally get the equation

$$ye^y + ry = u. \tag{5.8}$$

Notice that although u depends on r, s and t are arbitrary, so u can be considered as a variable independent of r. We therefore infer that equation (5.7) with two free parameters, and (5.8) with one free parameter are equivalent. For the solution(s) of the latter, we introduce the *r-Lambert function*, that is,

$$y = W_r(u).$$

In other words, the r-Lambert function at the point u is a real or complex number which satisfies the equation

$$W_r(u)e^{W_r(u)} + rW_r(u) = u.$$

Remember that the solution z of (5.7) equals $y + t$, hence, by plugging in the original values of r and u, we arrive at the correspondence between the $(1,1)$-type function $W\left(\begin{smallmatrix} t \\ s \end{smallmatrix}; w\right) = y + t = z$ and the r-Lambert function:

$$\boxed{W\left(\begin{matrix} t \\ s \end{matrix}; w\right) = t + W_{-e^{-t}w}\left(e^{-t}(t-s)w\right).} \tag{5.9}$$

This correspondence was first observed in [132]. It is now seen that it is enough to study the r-Lambert function which has only one parameter instead of the two-parameter $W\left(\begin{smallmatrix} t \\ s \end{smallmatrix}; w\right)$ function.

The following chapter will be dedicated to the study of the r-Lambert function.

5.4 A series representation for the $(1,1)$-type function

The Taylor series of $W\left(\begin{smallmatrix} t \\ s \end{smallmatrix}; w\right)$ around $w = 0$ is an interesting one:

$$\boxed{W\left(\begin{matrix} t \\ s \end{matrix}; w\right) = t - T\sum_{n=1}^{\infty} \frac{L_n'(nT)}{n} e^{-nt} w^n,} \tag{5.10}$$

where $T = t - s \neq 0$, and L_n' is the derivative of the nth *Laguerre polynomial* L_n. These polynomials can be defined via a *Rodrigues-type formula*:

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x}x^{n+\alpha}). \tag{5.11}$$

(In fact, these are the so-called *generalized Laguerre polynomials*; the classical ones are those with $\alpha = 0$.)

The proof of (5.10)

We use the Lagrange Inversion Theorem (1.51). Let

$$f(x) = e^x \frac{x-t}{x-s}.$$

We choose a point in which f is zero, and then we invert its series in this point. This point is t. Hence, by Lagrange's theorem,

$$W\left(\begin{matrix} t \\ s \end{matrix}; w\right) = t + \sum_{n=1}^{\infty} \frac{w^n}{n!} \lim_{\tau \to t} \frac{\mathrm{d}^{n-1}}{\mathrm{d}\tau^{n-1}} \left(\frac{\tau - t}{f(\tau)}\right)^n. \tag{5.12}$$

Here the only difficulty is the expression

$$\lim_{\tau\to t}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\tau^{n-1}}\left(\frac{\tau-t}{f(\tau)}\right)=\lim_{\tau\to t}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\tau^{n-1}}\left(\frac{\tau-s}{e^{\tau}}\right). \tag{5.13}$$

Recalling the Rodrigues formula (5.11), it can be seen that a modification of (5.13) will lead to a (generalized) Laguerre polynomial. Let us make this precise. The limit (5.13), if we expand it entirely, takes the form

$$e^{-nt}\sum_{k=1}^{n}A_{n,k}(-1)^{k-1}T^{k}. \tag{5.14}$$

Here $T=t-s$.

We are now going to determine the coefficients $A_{n,k}$. An easy calculation gives the following table:

$A_{n,k}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=1$	1						
$n=2$	2	2					
$n=3$	6	18	9				
$n=4$	24	144	192	64			
$n=5$	120	1200	3000	2500	625		
$n=6$	720	10800	43200	64800	38880	7776	
$n=7$	5040	105840	617400	1440600	1512630	705894	117649

These numbers cannot directly be found in the OEIS [4] but the row sum appears under the identification number A052885. There we can find that A052885 equals the sum of

$$(n-1)!n^{k-1}\binom{n}{k}\frac{1}{(k-1)!}.$$

Hence we can suspect that this is exactly what we are looking for,

$$A_{n,k}=(n-1)!n^{k-1}\binom{n}{k}\frac{1}{(k-1)!}.$$

Once we have this conjecture, the proof is easy (by induction). Hence we can step forward, substituting this into (5.12):

$$W\left(\begin{matrix}t\\ s\end{matrix};w\right)=t+\sum_{n=1}^{\infty}\frac{(we^{-t})^{n}}{n}\sum_{k=1}^{n}\frac{(-n)^{k-1}}{(k-1)!}\binom{n}{k}T^{k}=$$

$$W\left(\begin{matrix}t\\ s\end{matrix};w\right)=t-\sum_{n=1}^{\infty}\frac{(we^{-t})^{n}}{n^{2}}\sum_{k=1}^{n}\binom{n}{k}\frac{(-nT)^{k}}{(k-1)!}.$$

Recalling the explicit expression for the generalized Laguerre polynomials [167, p. 775, Table 22.3]:

$$L_n^{(\alpha)} = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!},$$

we can easily see that our inner sum in the Taylor series is simply

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-nT)^k}{(k-1)!} = -nTL_{n-1}^{(1)}(nT).$$

The relation [167, p. 778]

$$L_{n-1}^{(1)}(x) = -L_n'(x),$$

finalizes the proof. Note that $T \neq 0$ is necessary in the above argument, especially in (5.14). $T = 0$ stands for the standard log function. The proof is finished.

Further notes

1. **The origin of the generalized Lambert function.** According to our knowledge, it was a 2006 paper [162], where the idea of the generalization of the Lambert equation $xe^x = a$ to (5.3) first emerged. It was observed in [162] that a molecular physics problem [160] and the general relativistic two-body problem [122] both need the same generalization of W, which is, using our terminology, the $(2, 0)$-type Lambert W function.

2. **Quaternionic Lambert equation and Matrix Lambert equation.** The equation $qe^q = r$ where q and r are *quaternions* was studied in [65]. Equations of the form $S \exp(S) = A$, where S and A are *matrices* were studied in [38, 189], for example.

3. **Quadratic Lambert function.** Equations of the form $ze^{az^2+z} = w$ and their applications to general relativity are studied in [133].

4. **The discrete Lambert map and r-Lambert map.** The equation $xe^x = a$ can be studied over finite fields as well – in this case e must obviously be replaced with a *primitive root* g, and equality becomes congruence: $xg^x \equiv a \pmod{p^e}$ is the *discrete Lambert congruence*, and xg^x is the *discrete Lambert map*. This equation was extensively studied in [33, 117, 120, 190, 191]. The discrete r-Lambert map $xg^x + rx$ over finite fields was studied in [126].

5. **The p-adic Lambert function.** The p-adic Lambert function can formally be defined via the Taylor series for the principal branch. Some basic properties of the p-adic Lambert function were given in [127].

   See the web page of the author of this book for more information and up-to-date citations to newer papers on W and on its generalizations.

6. **Problems.** The general theory of the type (n, m) Lambert function might be elusive, but the next step after the well-known $(1, 1)$-type function (or, equivalently, the r-Lambert function) might be a good starting point. So one can ask for the $(2, 0)$, $(2, 1)$, and $(2, 2)$ type Lambert functions' branch structure, Riemann surface, asymptotics, and other properties.

   Another problem to study is the branch structure, Riemann surface, and asymptotics of the above-mentioned quadratic Lambert function.

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

6

The r-Lambert function

As it was noted in the previous chapter, the $(1,1)$-type Lambert function and the r-Lambert function W_r – the latter being the solution of the equation –

$$ze^z + rz = w, \tag{6.1}$$

– are equivalent. Since the r-Lambert function has only one free parameter, it is preferred to study the $(1,1)$-type Lambert function through W_r.

6.1 Equations solvable with the aid of W_r

We started the study of the classical W function in Chapter 1 by looking at some special values and some equations solvable by it. We do the same here; however, there are much less meaningful transformations that can be done on the equation (6.1).

6.1.1 Some special values of W_r

Values at zero

The classical Lambert function is zero at zero on the principal branch, and zero is a branch point for the other branches (see Figure 22). There is no such complication regarding the r-Lambert function when $r \neq 0$:

$$ze^z + rz = 0 \quad \Longrightarrow \quad z(e^z + r) = 0, \tag{6.2}$$

hence either

$$\boxed{W_r(0) = 0, \quad \text{or} \quad W_r(0) = \log(-r).}$$

Now that we are familiar with the notion of branches, we can infer that W_r has infinitely many branches, because (6.2) has infinitely many solutions, thanks to the multi-valuedness of the logarithm. The branch structure of the r-Lambert function will be studied in details later.

DOI: 10.1201/9781003168102-6

Special values by the inverse function

Since W_r is the inverse function of $ze^z + rz$, it follows that

$$\boxed{W_r(ze^z + rz) = z.}$$

For example, setting $z = 1$ and $z = -1$, we get

$$W_r(e + r) = 1, \quad \text{and} \quad W_r\left(-\frac{1}{e} - r\right) = -1, \tag{6.3}$$

respectively. Another example:

$$W_r(ee^e + re) = e.$$

Specializing further, the funny special value arises:

$$W_{e^e}(2ee^e) = e.$$

If $z = \log x$ in $ze^z + rz$, then

$$\log x \cdot e^{\log x} + r \log x = x \log x + r \log x = (x + r) \log x,$$

so

$$\boxed{W_r[(x + r) \log x] = \log x.}$$

Upon setting $x = e$ and $x = 1/e$, we get again (6.3).

An infinite class of special values

Observe that

$$\alpha e^\alpha + e^\alpha \alpha = 2\alpha e^\alpha,$$

hence, taking r to be e^α, we get that

$$\boxed{W_{e^\alpha}(2\alpha e^\alpha) = \alpha \quad (\alpha \in \mathbb{C}).}$$

In particular, for $\alpha = -1, 0, 1$

$$W_{1/e}(-2/e) = -1, \quad W_1(0) = 0, \quad W_e(2e) = 1.$$

Of course, each equality is valid only on a particular branch. Here we remark also that the W_r notation is not very fortunate, because the index r might be mistaken for the branch index of the classical W function. The context should always help in the clarification.

Transcendental values of W_r

We know that the values of the Lambert function are transcendental if its argument is an algebraic number, see p. 31.

A similar statement holds for the r-Lambert function, provided that r is an algebraic number:

For algebraic values of x and r, $W_r(x)$ is transcendental.

The proof is as follows. Let us suppose that $W_r(x)$ is algebraic for algebraic x and r. After rearranging the equation

$$W_r(x)e^{W_r(x)} + rW_r(x) = x,$$

we get

$$e^{W_r(x)} = \frac{x - rW_r(x)}{W_r(x)}.$$

The right-hand side is a rational expression of algebraic numbers, and so it is algebraic. But the Lindemann – Weierstrass theorem on p. 31 does not permit the exponential of an algebraic number (on the left-hand side) to be algebraic. This contradiction yields that $W_r(x)$ must be transcendental, as we stated.

6.1.2 The base change formula

If we want to solve the equation

$$zb^z + rz = w$$

for an arbitrary base $b \neq 0, 1$, we can do the following. Replace b^z with $e^{z \log b}$, and substitute $z \log b$ with, say, y. The resulting equation is

$$\frac{y}{\log b}e^y + r\frac{y}{\log b} = w, \quad \text{or} \quad ye^y + ry = w \log b.$$

This can be solved by W_r:

$$y = W_r(w \log b),$$

and thus

$$z = \frac{W_r(w \log b)}{\log b}.$$

We therefore have the following result:

$$\boxed{zb^z + rz = w \quad \Longrightarrow \quad z = \frac{W_r(w \log b)}{\log b}.}$$

6.1.3 An equation for the classical W function

The r-Lambert function helps to solve an interesting equation with the classical W function in it. Starting with the defining equation of W_r,

$$xe^x + rx = w, \tag{6.4}$$

so, by definition,

$$x = W_r(w). \tag{6.5}$$

Set

$$xe^x = z. \tag{6.6}$$

Then

$$x = W(z), \tag{6.7}$$

and by these last two equations, (6.4) becomes

$$z + rW(z) = w. \tag{6.8}$$

Putting (6.7) and (6.5) together,

$$W_r(w) = W(z).$$

We can find the solution for (6.8) by expressing z in the last displayed equation:

$$z = W^{-1}(W_r(w)) = W_r(w)e^{W_r(w)}.$$

Let us write out our finding explicitly:

$$\boxed{z + rW(z) = w \quad \Longleftrightarrow \quad z = W_r(w)e^{W_r(w)}.}$$

(The reverse statement is also true; our transformations were reversible.)

This result has a consequence which is certainly surprising at a first glance. Putting $z = 1$, we get that

$$1 + rW(1) = 1 + r\Omega \quad \Longrightarrow \quad 1 = W_r(1 + r\Omega)e^{W_r(1+r\Omega)}.$$

From here,

$$W_r(1 + r\Omega) = W(1) = \Omega,$$

hence

$$\boxed{W_r(1 + r\Omega) = \Omega \quad (r \in \mathbb{C}).}$$

We surely do not need this much hocus-pocus to get this result, however. We only need sharp eyes: if

$$we^w + rw = 1 + r\Omega,$$

then we can guess that $w = \Omega$, since $\Omega e^\Omega + r\Omega = 1 + r\Omega$ (for $\Omega e^\Omega = 1$). On the other hand, $w = W_r(1 + r\Omega)$, by definition.

6.2 The derivative and integral of W_r

In the first chapter we first studied the equations solvable by our new function, and got some special values of W; then we went on to determine the derivatives and integrals of this function. We follow this path here regarding the r-Lambert function. We therefore continue with the basic analytic properties of W_r.

6.2.1 The derivative of W_r

Taking the derivative on both sides of the equation

$$W_r(w)e^{W_r(w)} + rW_r(w) = w$$

with respect to the variable w, we have

$$W_r'(w)e^{W_r(w)} + W_r(w)e^{W_r(w)}W_r'(w) + rW_r'(w) = 1.$$

The derivative of the r-Lambert function is now expressible:

$$\boxed{W_r'(w) = \frac{1}{e^{W_r(w)}(1 + W_r(w)) + r}.} \tag{6.9}$$

6.2.2 Higher order derivatives

Recall that the higher order derivatives of the classical Lambert function are all expressible via a polynomial as it is shown in (1.33). The same is true for the r-Lambert function. Our statement is as follows.

$$\boxed{W_r^{(n)}(w) = \frac{p_{n,r}(W_r(w), e^{W_r(w)})}{(r + e^{W_r(w)}(1 + W_r(w)))^{2n-1}} \quad (n \geq 1).}$$

Here the bivariate polynomial $p_{n,r}(x, y)$ satisfies the recursion

$$p_{n+1,r}(x, y) = (r + y(1 + x))^2 p_{n,r}'(x, y) - (2n - 1)(2 + x)\, y\, p_{n,r}(x, y),$$

with the initial condition $p_{1,r}(x, y) = 1$. (In the recursion, the derivative is with respect to the variable w when $x = W_r(w)$, and $y = e^{W_r(w)}$.)

The proof

We proceed by induction. From (6.9) it follows that the theorem is true for $n = 1$ with $p_{1,r}(x, y) = 1$. Now let $n > 1$ be fixed and suppose that the statement is true for n. For the sake of brevity, we omit the argument w. Then for $W_r^{(n+1)}$ we have that it equals

$$\begin{aligned}
&= \left(\frac{p_{n,r}(W_r, e^{W_r})}{(r + e^{W_r}(1 + W_r))^{2n-1}} \right)' \\
&= \frac{1}{(r + e^{W_r}(1 + W_r))^{4n-2}} \left[p_{n,r}'(W_r, e^{W_r})(r + e^{W_r}(1 + W_r))^{2n-1} \right. \\
&\quad \left. -(2n - 1)(r + e^{W_r}(1 + W_r))^{2n-2}(e^{W_r}W_r'(1 + W_r) + W_r'e^{W_r})p_{n,r}(W_r, e^{W_r}) \right].
\end{aligned}$$

Substituting (6.9) in place of W_r' and doing some algebra, we obtain

$$W_r^{(n+1)} = \frac{p_{n+1,r}(W_r, e^{W_r})}{(r + e^{W_r}(1 + W_r))^{2n+1}}.$$

Low-degree examples

In particular, for $n = 2$, we have

$$W_r''(w) = \frac{-e^{W_r(w)}(2 + W_r(w))}{(r + e^{W_r(w)}(1 + W_r(w)))^3},$$

and

$$p_{2,r}(x, y) = -y(2 + x).$$

Moreover, for $n = 3$,

$$W_r'''(w) = \frac{e^{W_r(w)}[-r(3 + W_r(w)) + e^{W_r(w)}(9 + 8W_r(w) + 2W_r^2(w))]}{(r + e^{W_r(w)}(1 + W_r(w)))^5},$$

with

$$p_{3,r}(x, y) = y[-r(3 + x) + y(9 + 8x + 2x^2)].$$

6.2.3 Derivatives of $W_r(e^z)$

We are now going to describe the "r-version" of (1.39), the n-th derivative of the r-Lambert function composed with the exponential.

$$\boxed{W_r^{(n)}(e^z) = \frac{q_{n,r}(W_r(e^z), e^z)}{(e^z + e^z W_r(e^z) - rW_r^2(e^z))^{2n-1}} \quad (n \geq 1).}$$

The $q_{n,r}(x, y)$ polynomial satisfies the recursion

$$q_{n+1,r}(x, y) = \frac{1}{y}(rx^2 - y(1 + x))^2 q'_{n,r}(x, y)$$

$$-(2n - 1)(xy^2 - 2rx^2y + x^2y + 2xy + y - rx^3 - rx^2)q_{n,r}(x, y),$$

with the initial condition $q_{1,r}(x, y) = x$. (Here the derivative is understood to be referring to the variable z when $x = W_r(e^z)$, and $y = e^z$.)

For $n = 1$,

$$W_r'(e^z) = \frac{W_r(e^z)}{e^z + e^z W_r(e^z) - rW_r^2(e^z)},$$

and

$$q_{1,r}(x, y) = x.$$

For $n = 2$,

$$W_r''(e^z) = \frac{W_r(e^z)[r^2 W_r\,(e^z)^4 - re^z W_r\,(e^z)^3 + re^z\,(e^z - 1)\,W_r\,(e^z)^2 + e^{3z}]}{(e^z + e^z W_r(e^z) - rW_r^2(e^z))^3},$$

and

$$q_{2,r}(x, y) = x[r^2x^4 - rx^3y + rx^2y(y - 1) + y^3].$$

The proof of these is similar to the proof of the results in the previous section.

6.2.4 The derivative of W_r with respect to r

Considering the parameter r as a variable, it is possible to take the derivative of $W_r(w)$ with respect to this variable. The method is the same as for the variable w: we take the derivative of

$$W_r(w)e^{W_r(w)} + rW_r(w) = w$$

with respect to the variable r:

$$\begin{aligned}
&\frac{\partial}{\partial r}\left[W_r(w)e^{W_r(w)} + rW_r(w)\right] \\
&= e^{W_r(w)}\frac{\partial}{\partial r}W_r(w) + W_r(w)e^{W_r(w)}\frac{\partial}{\partial r}W_r(w) + W_r(w) + r\frac{\partial}{\partial r}W_r(w) \\
&= \frac{\partial}{\partial r}W_r(w)\left(e^{W_r(w)} + W_r(w)e^{W_r(w)} + r\right) + W_r(w).
\end{aligned}$$

On the right-hand side,

$$\frac{\partial}{\partial r}w = 0.$$

Hence

$$\boxed{\frac{\partial}{\partial r}W_r(w) = -\frac{W_r(w)}{e^{W_r(w)}(1 + W_r(w)) + r}.} \tag{6.10}$$

Notice that this result differs from (6.9) only in the minus sign and the nominator. We can express this observation this way:

$$\boxed{\frac{\frac{\partial}{\partial r}W_r(w)}{\frac{\partial}{\partial w}W_r(w)} = -W_r(w).}$$

6.2.5 The approximation of the r-Lambert function with the Lambert function

If the parameter r is small, the r-Lambert function should be close to the classical Lambert function. This qualitative remark can be made quantitative. Let us write $W_r(w)$ in the form of a Taylor series *with respect to* r around $r = 0$:

$$W_r(w) = c_0(w) + c_1(w)r + c_2(w)r^2 + \cdots . \tag{6.11}$$

The theory of Taylor series tells us that

$$c_n(w) = \frac{1}{n!}\frac{\partial^n}{\partial r^n}W_r(w)\bigg|_{r=0}.$$

Then it comes that

$$c_0(w) = W(w),$$

and by (6.10) it is similarly immediate that

$$c_1(w) = -\frac{W(w)}{e^{W(w)}(1+W(w))} = -\frac{W^2(w)}{w(1+W(w))}.$$

It can also be seen that

$$c_2(w) = -\frac{1}{2}\frac{W^3(w)(W^2(w)-2)}{w^2(W(w)+1)^3},$$
$$c_3(w) = -\frac{1}{6}\frac{W^4(w)(2W^4(w)-W^3(w)-12W^2(w)-6W(w)+6)}{w^3(W(w)+1)^5}.$$

These Taylor coefficients were deduced by G. Keady [105].

The series (6.11) thus becomes

$$\boxed{W_r(w) = W(w) - \frac{W^2(w)}{w(1+W(w))}r - \frac{1}{2}\frac{W^3(w)(W^2(w)-2)}{w^2(W(w)+1)^3}r^2 - \cdots.} \quad (6.12)$$

The importance of this series comes from the fact that if r is "small enough", the r-Lambert function can be approximated by the classical W function. The series can also be used to set the initial point for the numerical approximation algorithms for finding the values of W_r.

6.2.6 The integral of W_r and of its powers

The determination of the integral of the r-Lambert function and of its powers is a simple task because the integrals of W_r can be traced back to the evaluation of elementary integrals. Let

$$u = W_r(x). \quad (6.13)$$

Then

$$x = ue^u + ru,$$

whence

$$\mathrm{d}x = [e^u(1+u)+r]\mathrm{d}u.$$

Therefore

$$\int W_r(x)\mathrm{d}x = \int u[e^u(1+u)+r]\mathrm{d}u = \frac{r}{2}u^2 + e^u(1-u+u^2).$$

Using (6.13), we get our result:

$$\boxed{\int W_r(x)\mathrm{d}x = \frac{r}{2}W_r^2(x) + e^{W_r(x)}(1-W_r(x)+W_r^2(x)) + c.}$$

The integration of $W_r^n(x)$ for positive integer n can be done in a similar manner, but the calculations are more tedious. One gets that

$$\boxed{\begin{aligned}&\int W_r^n(x)\mathrm{d}x = \frac{r}{n+1}W_r^{n+1}(x)\\&+e^{W_r(x)}\left(W_r^{n+1}(x)+\sum_{k=-1}^{n-1}(-1)^{n-k}\frac{n\cdot n!}{(k+1)!}W_r^{k+1}(x)\right)+c\quad(n\geq 1).\end{aligned}}$$

Compare this result with (1.46).

For the moments of the r-Lambert function we have

$$\boxed{\int x^n W_r(x)\mathrm{d}x = \sum_{i=0}^{n}\binom{n}{i}r^{n-i}(F_{n,i}(W_r(x))-G_{n,i}(W_r(x))+c,}$$

where

$$F_{n,i}(u) = \frac{(-1)^n}{(1+i)^{n+3}}\left[\Gamma(n+r,-(1+i)W_r(x))-(1+i)\Gamma(n+2,-(1+i)W_r(x))\right],$$
$$G_{n,i}(u) = (-i)^{-n-2}\Gamma(n+2,-iu).$$

Here Γ is the incomplete Gamma function, see (1.43).

For instance,

$$\int xW_r(x)\mathrm{d}x = \frac{r^2}{3}u^3+re^u\left(u^3-u^2+2u-2\right)+\frac{1}{8}e^{2u}\left(4u^3-2u^2+2u-1\right)+c,$$

where (6.13) is to be applied.

6.3 The branch structure

6.3.1 The real values

Before analyzing the full picture of the branches of the r-Lambert function, we are going to see the real solutions of the equation (6.1).

Depending on the parameter $r\neq 0$, the r-Lambert function has one, two or three real branches and so (6.1) can have one, two or three real solutions. The full statement is the following.

1. If $r>1/e^2$, then $W_r:\mathbb{R}\to\mathbb{R}$ is a strictly increasing, everywhere differentiable function such that $\operatorname{sgn}(W_r(x))=\operatorname{sgn}(x)$. (The branch in question will later be denoted by $W_{r,0}$.)

2. If $r = 1/e^2$, then $W_r : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function which is differentiable on $\mathbb{R} \setminus \{-4/e^2\}$. Moreover, $\operatorname{sgn}(W_r(x)) = \operatorname{sgn}(x)$. (Later, when the complex branches will be taken into account, we shall be able to see that we actually have three branches here: $W_{1/e^2,0-}$, $W_{1/e^2,0+}$, and $W_{1/e^2,\pm 1}$.)

3. If $0 < r < 1/e^2$, then there are three branches of W_r containing real values in their range. We denote these by $W_{r,0-}$, $W_{r,0+}$, $W_{r,\pm 1}$. Let

$$\alpha_r = W_{-1}(-re) - 1, \quad \text{and} \quad \beta_r = W_0(-re) - 1,$$

   where W_{-1} and W_0 are the two branches of the classical Lambert function.

   Then α_r and β_r determine the above branches as follows:

   - $W_{r,0-} :]-\infty, f_r(\alpha_r)] \to]-\infty, \alpha_r]$ is a strictly increasing function,
   - $W_{r,\pm 1} : [f_r(\alpha_r), f_r(\beta_r)] \to [\alpha_r, \beta_r]$ is a strictly decreasing function, and
   - $W_{r,0+} : [f_r(\beta_r), +\infty[\to [\beta_r, +\infty[$ is a strictly increasing function.

   These three functions are differentiable on the interior of their domains.

4. Finally, if $r < 0$, then W_r has two real branches, $W_{r,0-}$ and $W_{r,0+}$. Let

$$\gamma_r = W(-re) - 1,$$

   where W is the classical Lambert function.

   Then for these branches we have that

   - $W_{r,0-} : [f_r(\gamma_r), +\infty[\to]-\infty, \gamma_r]$ is strictly decreasing, while
   - $W_{r,0+} : [f_r(\gamma_r), +\infty[\to [\gamma_r, +\infty[$ is a strictly increasing function.

   Both of them are differentiable in the interior of their domains.

Here sgn is the signum function such that $sgn(0) = 0$; and $f_r(x) = xe^x + rx$.

The proof

Proof. The proof is entirely elementary. What we have to take into account is that the derivative of $f_r(x) = xe^x + rx$ equals to zero exactly in $W(-re) - 1$:

$$f_r'(x) = e^x(1 + x) + r = 0 \Longrightarrow e^{1+x}(1 + x) = -re \Longrightarrow x = W(-re) - 1.$$

Then analyzing the possible values of this x we can easily confirm the number, domain and range of the different branches. The monotonicity and differentiability follows from the similar properties of $f_r = W_r^{-1}$. We shall see later why the point $W_{1/e^2}(-2)$ is exceptional.

6.3.2 Setting up the scene

Since the function $z\exp(z)+rz$ is a non-polynomial entire function, the defining equation of the r-Lambert function has infinitely many solutions, and all the solutions are of infinite multiplicity – as the general argument based on the Great Picard Theorem (see p. 40) states. Therefore, W_r has infinitely many branches. We are now going to study these branches and the Riemann surface of W_r. This task is a lot more complicated than the case when $r = 0$, since the branch structure strongly depends on r.

As an initial guess, we can suppose that the negative real line L is a good "approximation" for the branch cuts. Then we refine this idea during performing further analysis.

The branch points are determined by the analysis of the singularities of the derivative of W_r which was calculated in (6.9). The derivative $\partial/\partial x W_r(x)$ is singular when

$$e^{W_r(w)}(1 + W_r(w)) = -r.$$

This equation is of the form

$$e^x(1 + x) = a,$$

where $x = W_r(w)$, and $a = -r$. Performing the change of variable $y = 1 + x$, we have

$$e^{y-1}y = a,$$

or

$$ye^y = ea.$$

Now $y = W(ea)$, and $x = -1 + W(ea)$. Substituting the original values of a and x, we get the singular points of the derivative. These are

$$W_k(-re) - 1 \quad (k \in \mathbb{Z}).$$

Unlike for the classical Lambert function, here we have *infinitely many* finite branch points. We shall need their real and imaginary parts, so we define

$$\xi^*_\cdot = \Re(W_\cdot(-re) - 1), \tag{6.14}$$

$$\eta^*_\cdot = \Im(W_\cdot(-re) - 1). \tag{6.15}$$

The r-dependence will not be written out in order to keep the formulas simpler. We use dots in the formulas for now, because in different cases (ξ^*_k, η^*_k) will be defined by different branches of W. It is also important to remember that the indices of the numbers ξ^*_k and η^*_k will not be the same as the branch indices of W_r. This is an unavoidable complication, because, as r varies, branches appear and disappear, and it will not be possible to match all the indices all the time. In the general discussion of this subsection, we only write out those indices which always match.

We should keep in mind that

$$z_k^* := \xi_k^* + i\eta_k^* \tag{6.16}$$

is in the *range* of W_r. To get the corresponding points in the *domain*, we map these z_k^* numbers by $ze^z + rz$:

$$w_\cdot^* = z_\cdot^* e^{z_\cdot^*} + rz_\cdot^*. \tag{6.17}$$

By setting $w_k^* = a_k^* + ib_k^*$, we get

$$a_\cdot^* = \Re[(W_\cdot(-re) - 1)e^{W_\cdot(-re)-1} + r(W_\cdot(-re) - 1)], \tag{6.18}$$

$$b_\cdot^* = \Im[(W_\cdot(-re) - 1)e^{W_\cdot(-re)-1} + r(W_\cdot(-re) - 1)]. \tag{6.19}$$

Notice that if $r = 0$, then there is only one pair, $(\xi^*, \eta^*) = (-1, 0)$ on the *finite* complex plane belonging to the unique branch point of the classical Lambert function. For this point $w^* = -1/e$, and

$$W_{-1}(-1/e) = W_0(-1/e) = -1. \tag{6.20}$$

6.3.3 The branch separating curves in the z-plane

Let $z = s + it$ be a generic point in the z-plane. We shall see what equation must s and t satisfy if we want z to be on the curves separating the branches of W_r. Let $w = z\exp(z) + rz$, so that

$$\Re(w) = rs + e^s(s\cos t - t\sin t), \tag{6.21}$$

$$\Im(w) = rt + e^s(s\sin t + t\cos t). \tag{6.22}$$

(Compare these with (2.5)-(2.6).)

When we considered the branches of the logarithm, we cut up the plane along the negative real line L, so we made the restriction that $\Im(w) = 0$. This approach could be carried over to the analysis of branch cuts of the Lambert function, too. The only difficulty came from the fact that W has a branch point at $-1/e$, so the domains of W_{-1} and W_0 had to be cut up differently.

In the present case, the situation is more complicated. We know for certain that when $\Im(w) = b_k^*$, we have a branch point, and this point limits the branch cut. Later we analyze the mapping properties of W_r to see where to cut the planes. Thus we have the preliminary idea that, based on (6.22), we should analyze the equation

$$rt + e^s(s\sin t + t\cos t) = b_k^*. \tag{6.23}$$

These curves and therefore the branches have different characteristics depending on where the parameter r falls among the three intervals $[1/e^2, \infty[$, $]0, 1/e^2]$, and $]-\infty, 0[$. The case when $r = 1/e^2$ is special, too. Our discussion is therefore divided into four subsections.

6.3.3.1 The case when $r > 1/e^2$

The curves in the (s,t)-plane (z-plane) are plotted in Figure 32. It is seen that these curves determine one branch that extends to minus infinity. This branch will be denoted by $W_{r,0}$ and we call it the *principal branch*[1]. Moreover, (6.23) determines infinitely many branches (bounded from the left) that extend toward plus infinity. These will be denoted by $W_{r,k}$ where $k \in \mathbb{Z}_{\setminus 0} := \mathbb{Z} \setminus \{0\}$.

It is important to remark that no two of the non-principal branches meet on the finite plane (the neighboring branches are separated by positive distances), while the principal branch does meet with all the non-principal branches through a curve starting from the corresponding z_k^* of Figure 32.

Here we choose the indices of ξ_k^* and η_k^* such that, for $k > 0$,

$$\xi_{k+1}^* = \Re(W_k(-re) - 1),$$
$$\eta_{k+1}^* = \Im(W_k(-re) - 1),$$

and thus

$$z_{k+1}^* = W_k(-re) - 1.$$

While, for $k < 0$,

$$\xi_k^* = \Re(W_{k+1}(-re) - 1),$$
$$\eta_k^* = \Im(W_{k+1}(-re) - 1),$$

and

$$z_k^* = W_{k+1}(-re) - 1.$$

We keep z_0^* undefined until we will need it, namely, when $r < 0$.

We want to choose the branch cuts, which are presumably half-lines starting from the point w_k^*. There are two candidates: the half-line

$$J_{0,k} = \{x + iy \in \mathbb{C} : x \le a_k^*,\ y = b_k^*\} \quad (k \in \mathbb{Z}_{\setminus 0}),$$

or

$$L_{0,k} = \{x + iy \in \mathbb{C} : x \ge a_k^*,\ y = b_k^*\} \quad (k \in \mathbb{Z}_{\setminus 0}).$$

We are going to see how these half-lines are mapped by the different branches of W_r. The dashed line segments in Figure 32 on the left-hand-side of the points z_k^* belong to the image set $W_{r,0}(J_{0,k})$, while the dashed line segments on the right of z_k^* belong to $W_{r,k}(J_{0,k})$. The small circles in the figure symbolize the points z_k^*. Clearly, $W_{r,0}(J_{0,k}) \cap W_{r,k}(L_{0,k}) = \{z_k\}$.

These sets obviously do not separate the image sets of the branches, they are well inside the image of the corresponding branch. It is therefore seen that

[1]We try to keep as many analogies as possible with the classical Lambert function. However, here the principal branch has no analog in the $r = 0$ case, as we will see. When r tends to zero, this branch disappears and another branch emerges (already when $r = 1/e^2$) that we will call principal branch in that case. It is still natural to call $W_{r,0}$ principal because it contains the real line in its image, and also, its structure and shape are unique among all the branches.

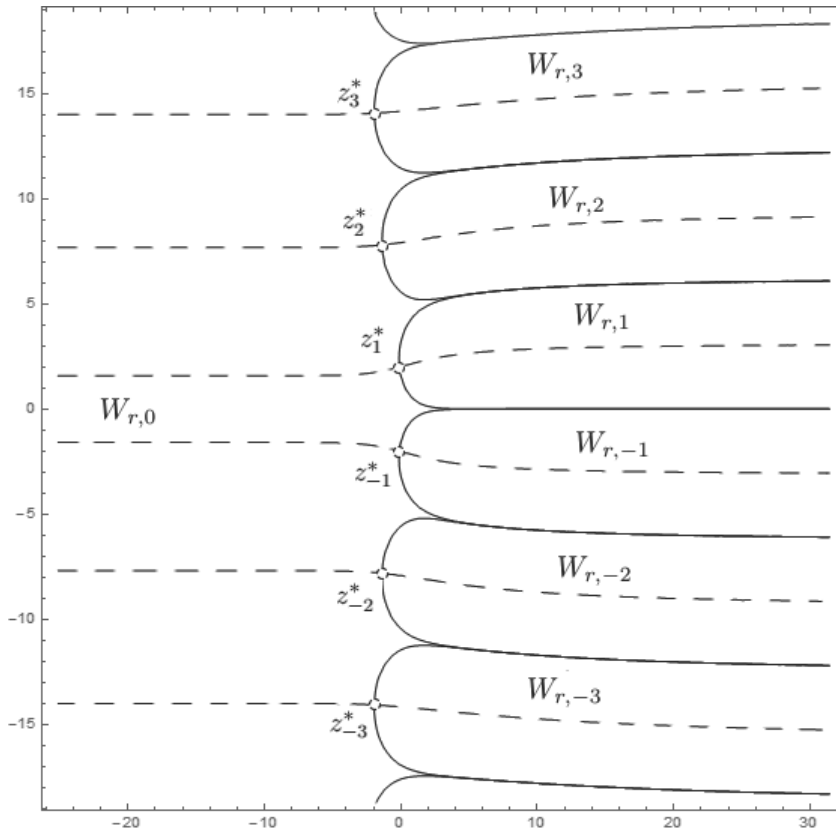


FIGURE 32
The curves in the z-plane that separate the image sets of the different branches of the r-Lambert function with $r > 1/e^2$ (for this plot, $r = 2$ was chosen). They are determined by equation (6.23). This plot does not contain the closure information.

the branch cuts are chosen to be the lines

$$L_{0,k} = \{x + iy \in \mathbb{C} : x \geq a_k^*,\, y = b_k^*\} \quad (k \in \mathbb{Z}_{\setminus 0}) \tag{6.24}$$

in the w-plane, and the corresponding branch points are

$$w_k^* := a_k^* + ib_k^*.$$

We hence cut up the domain of the principal branch $W_{r,0}$ along *all* the lines $L_{0,k}$, and we take a copy $\mathbb{C}_k$ of $\mathbb{C}$ for each of the non-principal branches, cut $\mathbb{C}_k$ up along its corresponding $L_{0,k}$ line, and glue the cuts of these planes to the corresponding cuts of $\mathrm{dom}(W_{r,0})$, as it is seen on Figure 33. The way

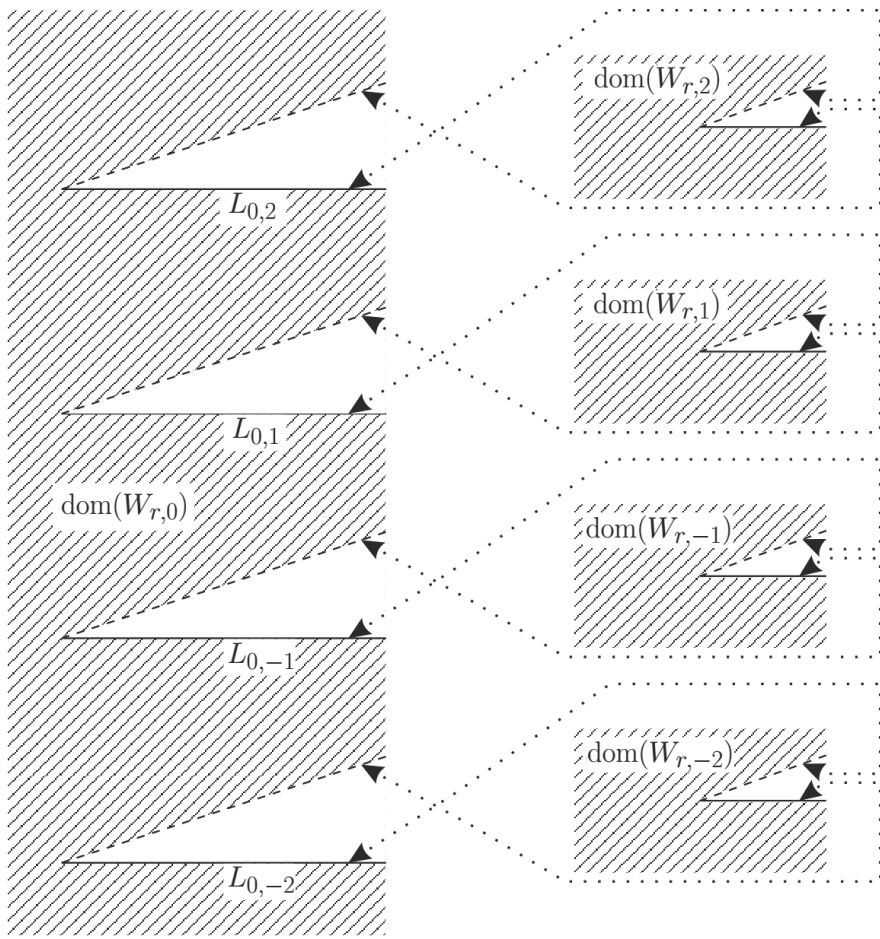


FIGURE 33
The Riemann surface of the r-Lambert function for an arbitrary but fixed $r > 1/e^2$ parameter. The dashed lines show that the plane is open along it. The dotted lines with arrows show the way the branch cuts are glued. The lines $L_{0,k}$ are defined in (6.24).

of gluing is determined so that the usual requirement of *counter-clockwise continuity* (CCC) holds. Indeed, drawing a small closed curve in the counter-clockwise direction around a branch point $w_k^* \in \mathrm{dom}(W_{r,0})$, we eventually reach the line $L_{0,k}$ where the set $\mathrm{dom}(W_{r,0})$ is closed, then we jump to the sheet of $\mathrm{dom}(W_{r,k})$, we run around the point $w_k^* \in \mathrm{dom}(W_{r,k})$, reaching the line $L_{0,k} \subset \mathrm{dom}(W_{r,k})$, and we then arrive onto the set $\mathrm{dom}(W_{r,0})$ again. This shows, in particular, that the monodromy group at the branch point w_k^*

is C_2, the cyclic group of order two ($k \in \mathbb{Z}_{\setminus 0}$). The Riemann surface of W_r for $r > 1/e^2$ is therefore constructed.

The branch closure

The above description of the Riemann surface helps us see where the ranges of different branches of W_r are closed and where they are open.

By the above analysis, we infer that for all $k \in \mathbb{Z}_{\setminus 0}$, the boundary curve in the z-plane *above* the point z_k^* belongs to the image set of $W_{r,k}$, while the curve *below* z_k^* belongs to $W_{r,0}$. Clearly, as w_k^* is a branch point, $z_k^* = W_{r,0}(w_k^*) = W_{r,k}(w_k^*)$. See Figure 34 for a visual guide.

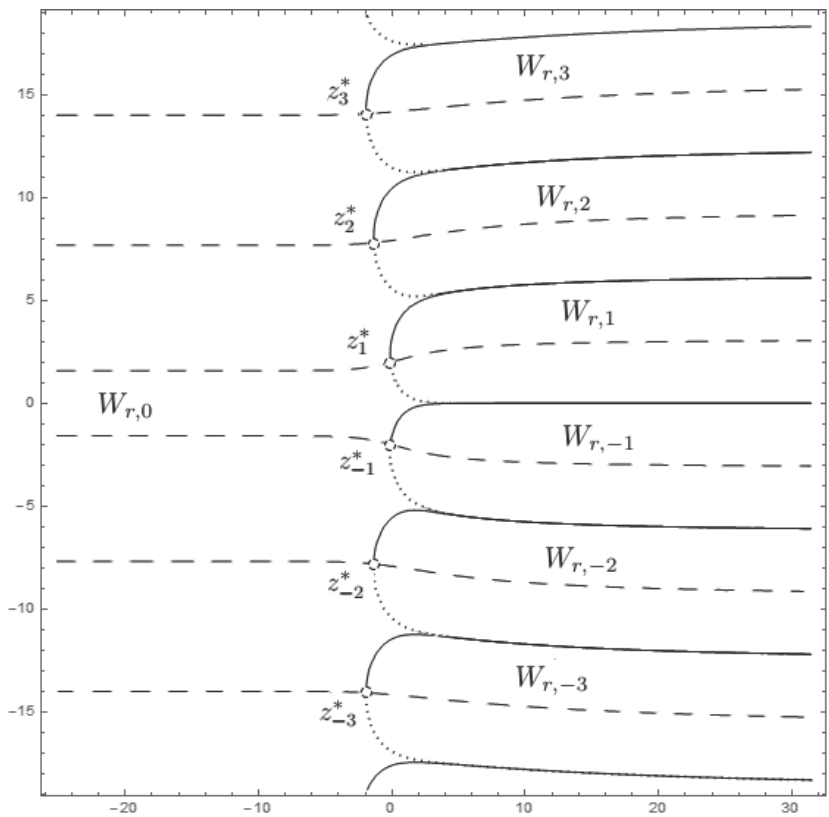


FIGURE 34
Branch closures in the case when $r > 1/e^2$.

Let us see this in an example when, say, $k = 2$. Taking the line $L_{0,2}$, we see in Figure 33 that the sheets for the principal branch and for $W_{r,2}$ are glued together in such a way that $L_{0,2}$ itself still belongs to $W_{r,0}$ by CCC, thus the image of $L_{0,2}$ belongs to the range of $W_{r,0}$.

To see these in more details, we analyze the curve

$$\Gamma := \{W_{r,0}(x + ib_2^*) : x \in \mathbb{R}\}$$

in the z-plane. It splits into two disjoint parts:

$$\Gamma = \Gamma_1 \cup \Gamma_2,$$

where

$$\Gamma_1 = W_{r,0}(L_{0,2}),$$
$$\Gamma_2 = W_{r,0}(\mathbb{R} \setminus L_{0,2}).$$

The curve Γ_2 is the *dashed* curve which, as x comes from minus infinity toward a_2^*, comes from minus infinity until the point z_2^* in Figure 34 (z_2^* itself does not belong to Γ_2); while Γ_1 is the *dotted* curve which runs from the point z_2^* downward and rightward in the image of $W_{r,0}$ as x goes from a_2^* toward plus infinity. Thus the range of $W_{r,0}$ is closed, but the range of $W_{r,2}$ is open along the curve $\Gamma_1 \setminus \{z_2^*\}$. Since w_2^* is a branch point,

$$W_{r,0}(w_2^*) = W_{r,2}(w_2^*) = z_2^*.$$

Next, let us see how the same sets are mapped by $W_{r,2}$. We define the curve

$$\Delta = \{W_{r,2}(x + ib_2^*) : x \in \mathbb{R}\}$$

which also splits into two disjoint parts:

$$\Delta = \Delta_1 \cup \Delta_2,$$

where

$$\Delta_1 = W_{r,2}(L_{0,2}),$$
$$\Delta_2 = W_{r,2}(\mathbb{R} \setminus L_{0,2}).$$

The curve Δ_2 is the dashed line coming from plus infinity as we run with x from minus infinity, and reaches z_2^* when $x = a_2^*$ (Δ_2 itself does not contain z_2^*, this point already belongs to Δ_1). When x grows further, we are on Δ_1, going upward and to the right from z_2^* on the solid line above z_2^*.

According to the previous description, in Figure 34 the dotted lines belong to the principal branch, while the solid lines belong to the branches below them. Note that the dotted curves go toward plus infinity, but on the figure they seem to be merged to the solid lines. This is only virtual as the numerical difference between these points is tiny; the dotted curves and the solid ones *do*

not meet in the finite plane. For example, the real line maps onto the real line by $W_{r,0}$ (not presented in Figure 34) and this set in the range of the principal branch lies between $W_{r,1}$ and $W_{r,-1}$ so these two branches are separated by $\mathbb{R}$, they do not meet in the finite plane.

It is still not apparent, but the closures are well chosen in the sense that in the limit $r \to 0$ we get the closure structure of the branches of the classical Lambert function. This will be more clear in the following sections when we study the case $0 < r \leq 1/e^2$.

Some additional properties of the principal branch $W_{r,0}$

For $r > 1/e^2$, the principal branch $W_{r,0}$ of the r-Lambert function contains the whole real line in its image (the real line is mapped onto itself by this branch), and this branch is unbounded from the left and from the right on the complex plane. Horizontal lines $x + iy$ ($x \in \mathbb{R}$, y fixed) maps by $W_{r,0}$ to curves from minus infinity to plus infinity such that if

$$0 < y < b_1^*,$$

then the curve runs above the s-axis but below the first dashed line in the upper half-plane (see on Figure 35). As x grows, the curve

$$\{W_{r,0}(x + iy) \mid x \in \mathbb{R}\}$$

runs into the gap between $W_{r,-1}$ and $W_{r,1}$.

If, in turn, y is negative such that

$$0 > y > b_{-1}^*,$$

then $\{W_{r,0}(x + iy) \mid x \in \mathbb{R}\}$ runs below the s-axis, and it penetrates the area between the s-axis and the branch $W_{r,-1}$.

Moreover, if y is a fixed real number in the interval

$$b_k^* < y < b_{k+1}^*,$$

with a given $k > 0$, then the curve $\{W_{r,0}(x + iy) \mid x \in \mathbb{R}\}$ runs between two consecutive dashed lines in the upper half-plane. As x grows, this curve penetrates into the gap between $W_{r,k}$ and $W_{r,k+1}$.

If for some $k < 0$

$$b_k^* > y > b_{k-1}^*, \tag{6.25}$$

then the image of $x+iy$ by the principal branch runs between two consecutive dashed lines in the lower half-plane such that it runs toward the area between $W_{r,-k}$ and $W_{r,-k-1}$.

A typical situation for $y > 0$ is shown in Figure 35, where we have chosen $y = y_\pm$ to be $b_1^* \pm 1$ when $r = 2$. The lower dotted curve in the image of the principal branch corresponds to y_-, while the higher dotted curve in $W_{r,0}$ belongs to y_+. The other two dotted curves are the images of the same set $x + iy$ by $W_{r,1}$.

Finally, the situation when y equals to some b_k^* was detailed above.

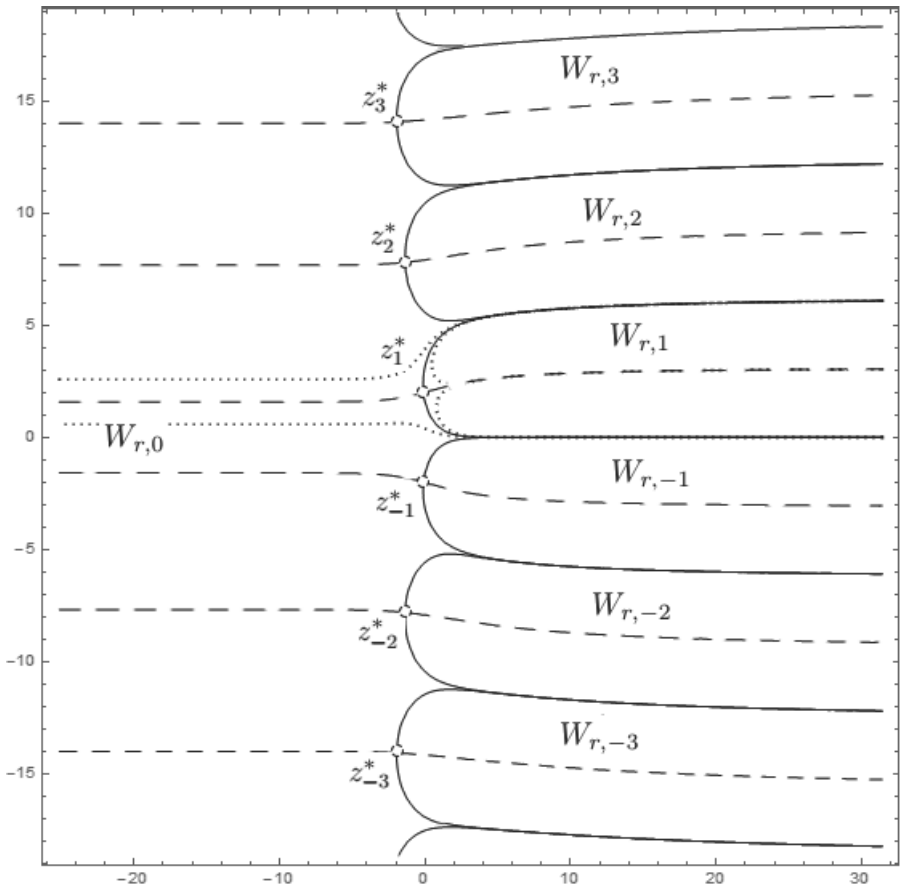


FIGURE 35
Images of some horizontal lines in the case when $r > 1/e^2$. The meaning of the dotted lines is explained after (6.25).

6.3.3.2 The case when $r = 1/e^2$

When r tends to $1/e^2$ from the right, an interesting phenomenon occurs:

$$(\xi_1^*, \eta_1^*) \to (-2, 0),$$
$$(\xi_{-1}^*, \eta_{-1}^*) \to (-2, 0),$$

so $z_1^* = z_{-1}^* = -2$. Geometrically (see Figure 36), the branch curves that previously formed the boundaries of $W_{r,1}$ and $W_{r,-1}$ now share the point -2 which belongs to the real pre-image $-4/e^2$. That is, the two, previously (i.e., when $r > 1/e^2$) disjoint branches $W_{r,1}$ and $W_{r,-1}$ now form a *new branch*

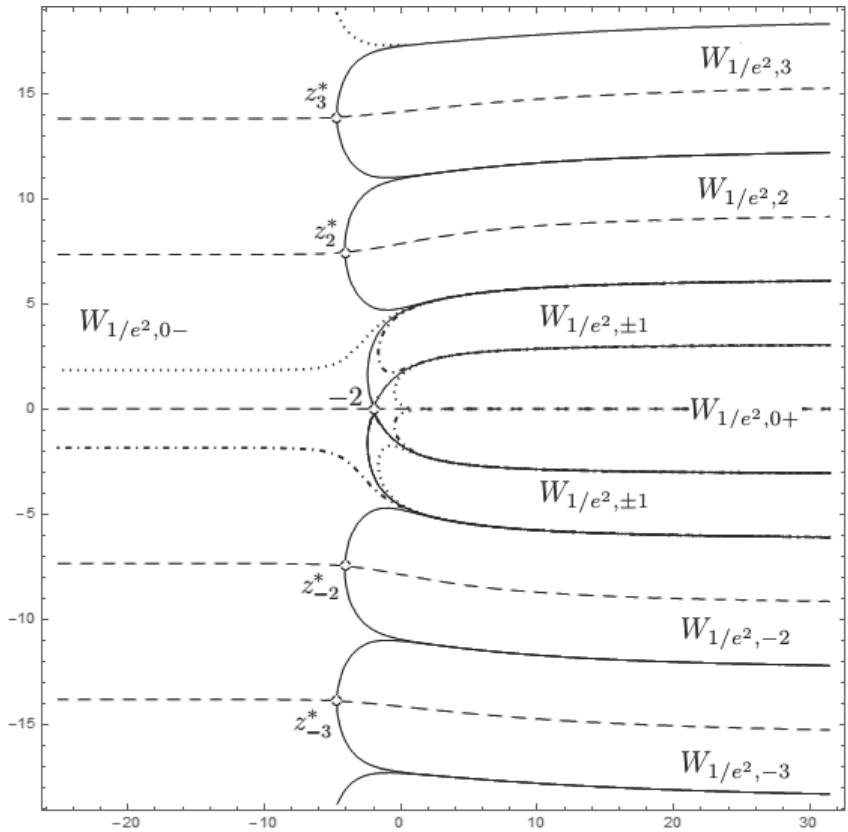


FIGURE 36
The image sets of the different branches for $r = 1/e^2$ without closure information. The dotted and dash-dotted curves are explained in the next subsection.

which will be denoted by $W_{1/e^2,\pm1}$. This branch contains one real point:

$$W_{1/e^2,\pm1}\left(-\frac{4}{e^2}\right) = -2.$$

In other words,

$$w_{-1}^* = w_1^* = -\frac{4}{e^2}. \tag{6.26}$$

By further analysis of Figure 36, it is seen that the branch that was formerly denoted by $W_{r,0}$ will contain real numbers now only from minus infinity up to -2. We rename this branch as $W_{r,0-}$.

The remaining set of reals, $[-2, +\infty[$, belongs to another *new branch*, denoted by $W_{r,0+}$. Summarizing we know, we can write

$$\begin{aligned}
&\mathrm{ran}(W_{1/e^2,\pm1}) \cap \mathbb{R} = \{-2\},\\
&\mathrm{ran}(W_{1/e^2,0-}) \cap \mathbb{R} =]-\infty, -2],\\
&\mathrm{ran}(W_{1/e^2,0+}) \cap \mathbb{R} = [-2, \infty[\,,
\end{aligned}$$

where $\mathrm{ran}(f)$ is the *range* of the function f. We thus have the following:

$$W_{1/e^2,\pm1}\left(-\frac{4}{e^2}\right) = W_{1/e^2,0-}\left(-\frac{4}{e^2}\right) = W_{1/e^2,0+}\left(-\frac{4}{e^2}\right) = -2. \tag{6.27}$$

For the other $W_{r,k}$ branches (when $|k| > 1$), there is no change in their character when $r \to \frac{1}{e^2}+$, apart from the changes in the w_k^* and the corresponding z_k^* values. Hence, for $|k| > 1$, the branch cuts are (taking into account the renaming of $W_{1/e^2,0-}$):

$$L_{0-,k} = \{x + iy \in \mathbb{C} : x \geq a_k^*,\, y = b_k^*\} \quad (|k| > 1), \tag{6.28}$$

as for the case when $r > 1/e^2$.

The interval $[-4/e^2, +\infty[$ is mapped onto the interval $]-2, \infty[$ by $W_{1/e^2,0+}$ which is inside the range. Thus, we choose the branch cut for this branch to be

$$L_{0+,\pm1} = \left]-\infty, -\frac{4}{e^2}\right] .$$

Since the image set of this branch is surrounded by the image of the branch $W_{1/e^2,\pm1}$, we also cut its sheet at the line $L_{0+,\pm1}$. This is why the index of this cut is chosen as it is.

We still need to know how to cut the sheet of $W_{1/e^2,0-}$. For this branch we choose

$$L_{0-,\pm1} = \left[-\frac{4}{e^2}, +\infty\right[.$$

The sheet of $W_{1/e^2,0-}$ must be glued to the sheet of $W_{1/e^2,\pm1}$, so we cut up the sheet of this latter branch along $L_{0-,\pm1}$.

All the branch cuts are determined, the result is visualized in Figure 37.

The branch closures are chosen such that CCC is satisfied, and also such that when $r \to 0+$ they are in agreement with the standard branch closure of the W function. Regarding the image sets, this means the following. The upper part of $W_{1/e^2,\pm1}$ till the point -2 belongs to $W_{1/e^2,\pm1}$, while the part below -2 limited by the dotted curve in Figure 5 belongs to $W_{1/e^2,0-}$. Moreover, the upper limiting curve of $W_{1/e^2,0+}$ belongs to $W_{1/e^2,0+}$ while the lower limiting curve belongs to $W_{1/e^2,\pm1}$ instead. See Figure 38 for the illustration.

By the chosen branch closure, we have the following facts, too:

$$W_{1/e^2,0-}(\mathbb{R}) =$$

$$[-\infty, -2] \cup \{\text{the limiting dotted curve of the lower part of } W_{1/e^2,\pm1}\}.$$

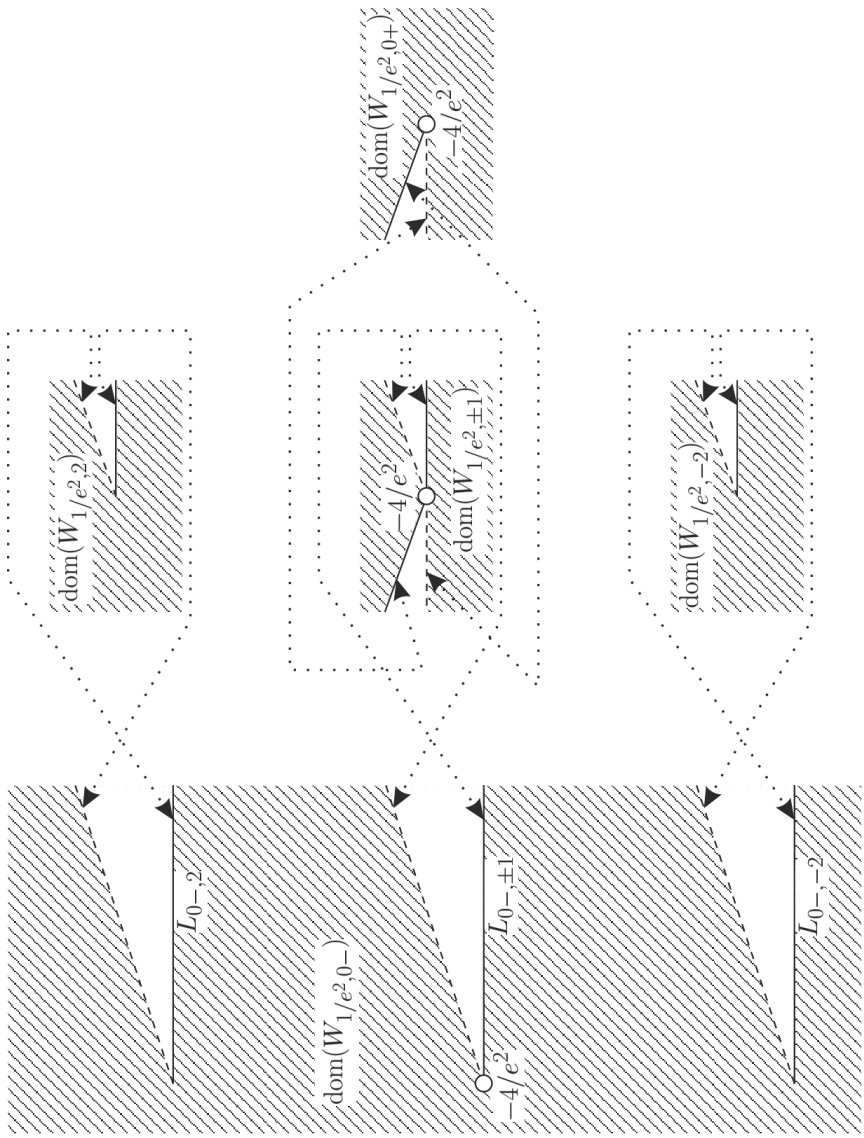


FIGURE 37
The construction of the Riemann surface of the $1/e^2$-Lambert function. The solid line going "north-west" from the point $-4/e^2$ in the domain of $W_{1/e^2,\pm1}$ is the line $L_{0-,\pm1}$, while the line which goes to the east from $-4/e^2$ (and the solid line going "north-west" in dom($W_{1/e^2,0+}$)) is $L_{0+,\pm1}$.

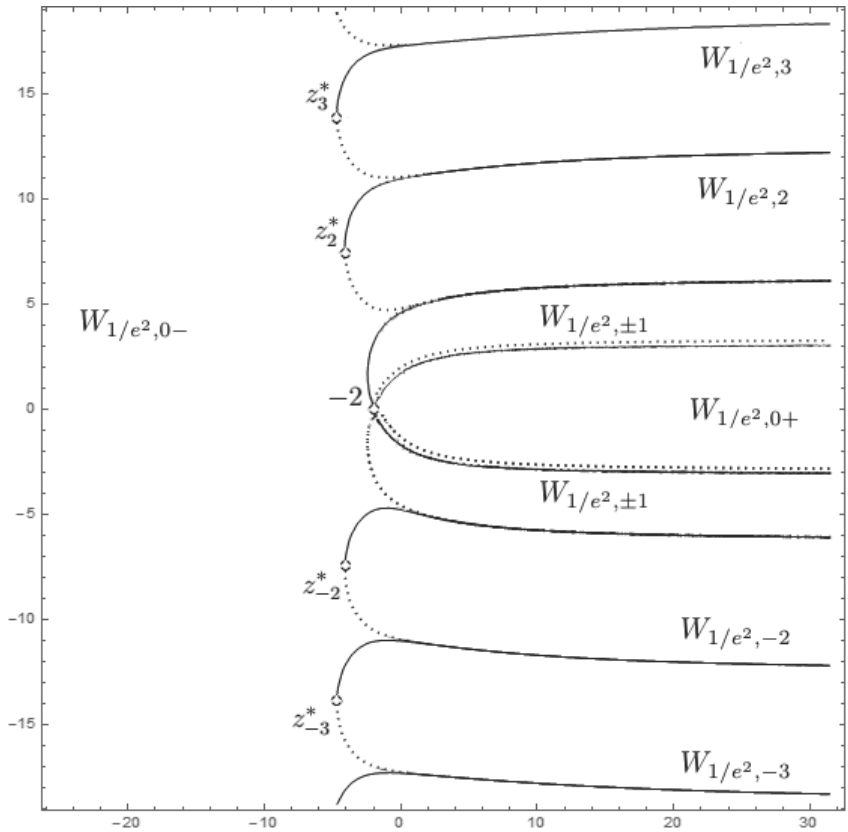


FIGURE 38
The closure information on the image sets of the different branches of the $1/e^2$-Lambert function.

Moreover,

$$W_{1/e^2,0+}(\mathbb{R}) =$$

$$[-2, +\infty[\cup \{\text{the solid line running upwards from } -2 \text{ in } W_{1/e^2,0+}\}.$$

Here the curves refer to those in Figure 38.

Some illustrative mapping properties of $W_{1/e^2,0-}$, $W_{1/e^2,0+}$ and $W_{1/e^2,\pm1}$

For a deeper understanding of how the different branches of W_{1/e^2} map the points in their domain to the z-plane, we make the following analysis.

We specialize (6.14)-(6.16) as follows:

$$z^*_{k+1} = W_k(-re) - 1 \quad (k > 0),$$
$$z^*_k = W_k(-re) - 1 \quad (k < 0).$$

(The value of z^*_0 will be defined later.) With this choice, we agree with $z^*_{-1} = z^*_1 = -2$; furthermore,

$$z^*_2 = -4.08884 + 7.46149i,$$
$$z^*_{-2} = \overline{z^*_2}.$$

(These values can roughly be seen in the plot of Figure 36.) This and (6.17) yield that

$$w^*_2 = -0.69511 + 0.994319i,$$
$$w^*_{-2} = \overline{w^*_2}.$$

By using our notation

$$w^*_k = a^*_k + ib^*_k,$$

we have, in particular, that

$$b^*_{\pm 2} = \pm 0.994319.$$

We choose a line with an imaginary part being $1/2$ and see how it is mapped by the mentioned three branches of the $1/e^2$-Lambert function. Since $0 < 1/2 < b^*_2$, we know without any calculation that the said line will be mapped by $W_{1/e^2,0-}$ to a curve *below* the dotted curve of Figure 36. The left part of the dotted curve, we remember, is $W_{1/e^2,0-}(\mathbb{R} + ib^*_2)$.

The curve

$$W_{1/e^2,0-}\left(\mathbb{R} + \frac{1}{2}i\right)$$

is the dotted line in Figure 36 inside the range of the branch. Moreover, the curves

$$W_{1/e^2,0+}\left(\mathbb{R} + \frac{1}{2}i\right), \quad \text{and} \quad W_{1/e^2,\pm 1}\left(\mathbb{R} + \frac{1}{2}i\right)$$

are also depicted in Figure 36 inside the corresponding domains.

It is seen that $W_{1/e^2,0-}$ and $W_{1/e^2,0+}$ map the upper half-plane to the upper half of their image set, and the lower half-plane is mapped to the lower half of their range. To the contrary, $W_{1/e^2,\pm 1}$ maps the upper half-plane to the lower part of its range, and vice versa.

The dash-dotted curves of Figure 36 are the corresponding sets

$$W_{1/e^2,0-}\left(\mathbb{R} - \frac{1}{2}i\right), \quad W_{1/e^2,0+}\left(\mathbb{R} - \frac{1}{2}i\right), \quad \text{and} \quad W_{1/e^2,\pm 1}\left(\mathbb{R} - \frac{1}{2}i\right).$$

Let us now determine the monodromy group at the branch point $-4/e^2$ (see also Figure 37). Let us choose a curve around this point which starts, say, in the upper half of the domain of $W_{1/e^2,\pm 1}$. As the curve goes counter-clockwise, it will eventually jump up to $W_{1/e^2,0+}$, goes around $-4/e^2$, and jumps onto the sheet of $W_{1/e^2,\pm 1}$ again, but now it lands on the lower half. After making a half-circle here, the curve jumps up onto the sheet of $W_{1/e^2,0-}$, goes around the branch point, then hops on the upper half of the sheet of $W_{1/e^2,\pm 1}$, making a full closed path. It is seen that the monodromy is therefore the cross-product of two cyclic groups of order two:

$$C_2 \times C_2.$$

6.3.3.3 The case when $0 < r < 1/e^2$

If we further decrease r from $1/e^2$, the main change in the branch structure is that the point -2 in the z-plane blows up and becomes an interval of positive length. In place of (6.26) we now have

$$I = [z^*_{-1}, z^*_1] = [W_{-1}(-re) - 1, W_0(-re) - 1]. \tag{6.29}$$

(Here W_{-1} and W_0 are the two real branches of the classical Lambert function.)

The pre-images of the end-points of I by the r-Lambert function are $w^*_{-1} = a^*_{-1} + ib^*_{-1}$ and $w^*_1 = a^*_1 + ib^*_1$ in the w-plane. These points are, in fact, real, hence $b^*_{-1} = b^*_1 = 0$. Furthermore, we can express a^*_{-1} and a^*_1 explicitly by recalling (6.18)-(6.19):

$$\begin{aligned} w^*_{-1} = a^*_{-1} &= z^*_{-1} \exp(z^*_{-1}) + r z^*_{-1}, \\ w^*_1 = a^*_1 &= z^*_1 \exp(z^*_1) + r z^*_1. \end{aligned}$$

Notice that, by the mapping properties of the Lambert function,

$$a^*_1 < a^*_{-1},$$

thus we introduce the interval

$$J = [a^*_1, a^*_{-1}]$$

in the w-plane. The role of these points is that if $w > a^*_{-1}$ is a real number, then $W_{r,0-}(w)$ becomes complex; if $w \in J$, then $W_{r,\pm 1}(w)$ is real; and for $w < a^*_1$ real, then $W_{r,0+}(w)$ becomes complex. Thus it is apparent that the branch cuts should be chosen as follows.

For the branches $W_{r,0-}$ and $W_{r,\pm 1}$, the branch cut is

$$L_{0-,\pm 1} = [a^*_{-1}, +\infty[\,,$$

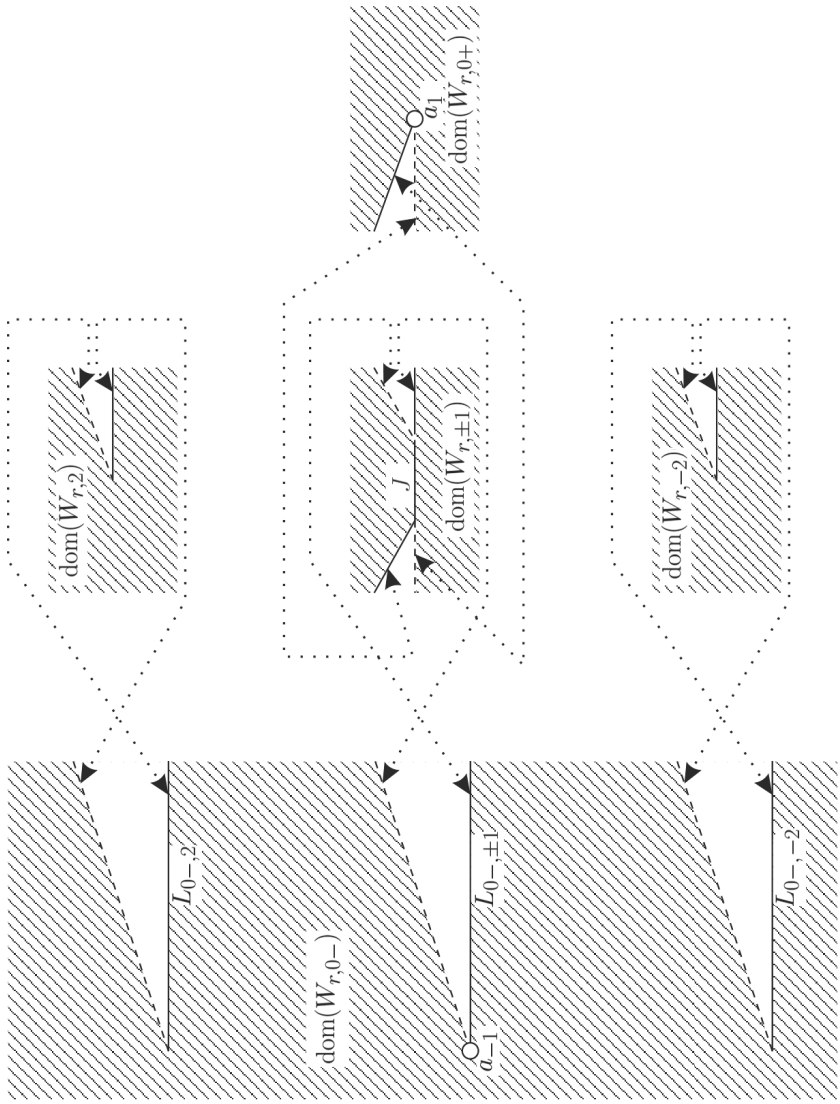


FIGURE 39
The construction of the Riemann surface of the r-Lambert function when $0 < r < 1/e^2$.

while for the $W_{r,\pm1}$ and $W_{r,0+}$ branches, it is

$$L_{0+,\pm1} = [-\infty, a_1^*] .$$

For the arising Riemann surface, see Figure 39.

If $|k| > 1$, no essential change happens, (6.28) is valid. The monodromy around all the branch points is simply C_2.

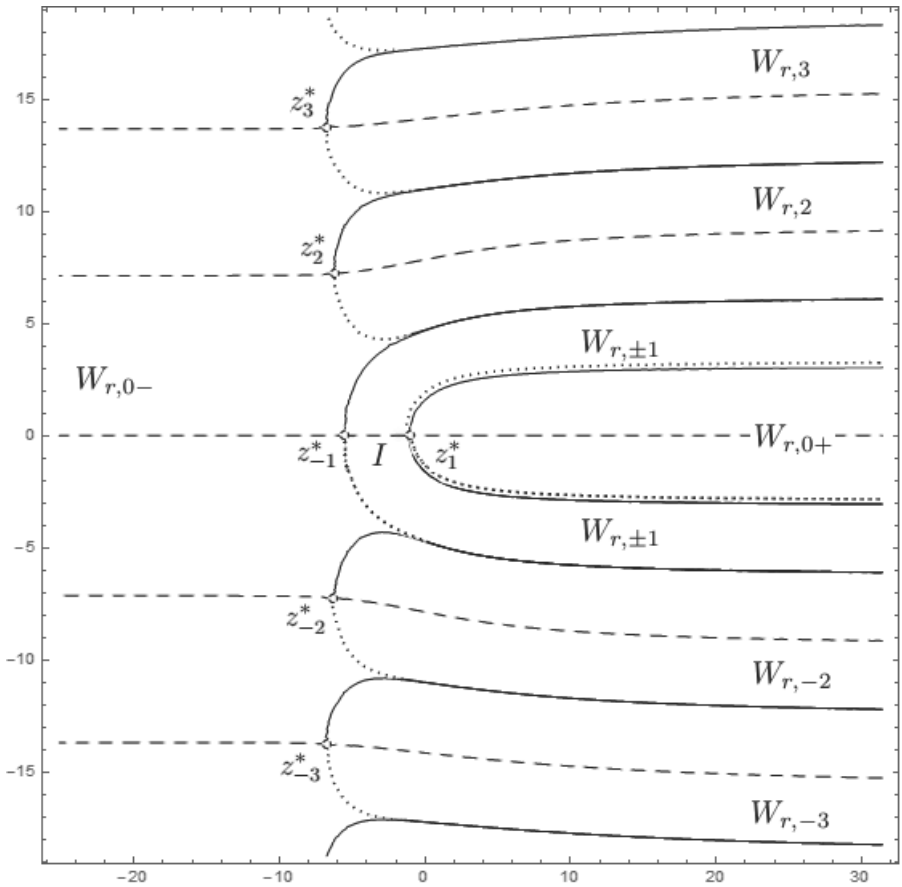


FIGURE 40
The image sets of the different branches of the r-Lambert function when $0 < r < 1/e^2$, together with closure information.

The branch closure can be deduced from the case when $r = 1/e^2$, see the illustration of the image sets in Figure 40.

6.3.3.4 The case when $r < 0$

We have seen how to construct the Riemann surface of the r-Lambert function, and how the image sets are distributed on the complex plane when $r > 0$. When $r = 0$, these determinations were done in Chapter 2. We still need to deal with the case $r < 0$.

When $r < 0$ and $k \neq 0$, the $W_{r,k}$ branches are bounded from the left again by the quantities z_k^* as they were for $r > 1/e^2$. Now we go back to the

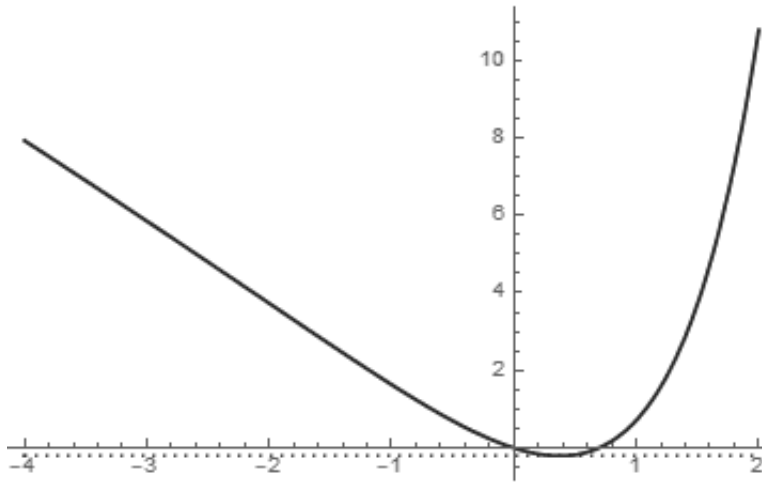


FIGURE 41
The function $ze^z + rz$ maps the interval $]-\infty, z_0^*]$ to the interval $[w_0^*, +\infty[$, but $[z_0^*, +\infty[$ is mapped onto the same interval, too. The dotted horizontal line is at $y = w_0^*$.

non-shifted indices again:

$$z_k^* = \xi_k^* + i\eta_k^* = W_k(-re) - 1.$$

The importance of this will be clear soon.

If $r < 0$, the number $\eta_0^* = \Im(W_0(-re) - 1)$ is always 0 ($W_0(w)$ is real for real non-negative w), so there will be a real branch point at

$$z_0^* = \xi_0^* = W_0(-re) - 1.$$

(This is why we have spared z_0 from being used so far.) The image of the real line thus belongs to two branches that we denote by $W_{r,0-}$ and $W_{r,0+}$. A part of the real line maps to two half-lines: the interval $[w_0^*, +\infty[$ maps to $[-\infty, z_0^*]$ by $W_{r,0-}$, while the same interval maps to $[z_0^*, +\infty]$ by $W_{r,0+}$. These observations can read out from the plot of Figure 41.

The union of the two image sets $W_{r,0-}([w_0^*, +\infty[)$ and $W_{r,0+}([w_0^*, +\infty[)$ forms the horizontal dashed line at z_0^* in Figure 42.

The half-line

$$L_{0-,0+} = [-\infty, a_0^*] \tag{6.30}$$

(where $a_0^* = \Re(w_0^*)$) is therefore chosen to be the branch cut in the domain of $W_{r,0-}$ and of $W_{r,0+}$.

To illustrate the complete picture of how the real line is mapped by these two branches, let us introduce the two curves

$$\Gamma_{0,\pm} := \{W_{r,0\pm}(x) : x \in \mathbb{R}\}.$$

Both of these have two parts:

$$\Gamma_{0,\pm} = \Gamma_{0,\pm}^{(1)} \cup \Gamma_{0,\pm}^{(2)},$$

where

$$\Gamma_{0,\pm}^{(1)} = \{W_{r,0\pm}(x) : x \in]-\infty, w_0^*]\} = W_{r,0\pm}(L_{0-,0+}),$$

and

$$\Gamma_{0,\pm}^{(2)} = \{W_{r,0\pm}(x) : x \in [w_0^*, +\infty[\,\}.$$

As we have already said, $\Gamma_{0,\pm}^{(2)}$ is the interval $[-\infty, z_0^*]$ if the minus sign is taken, and it is $[z_0^*, +\infty[$ if the plus sign is used. Hence, again, $\Gamma_{0,-}^{(2)} \cup \Gamma_{0,+}^{(2)} = \mathbb{R}$, the dashed line crossing z_0^* in Figure 42.

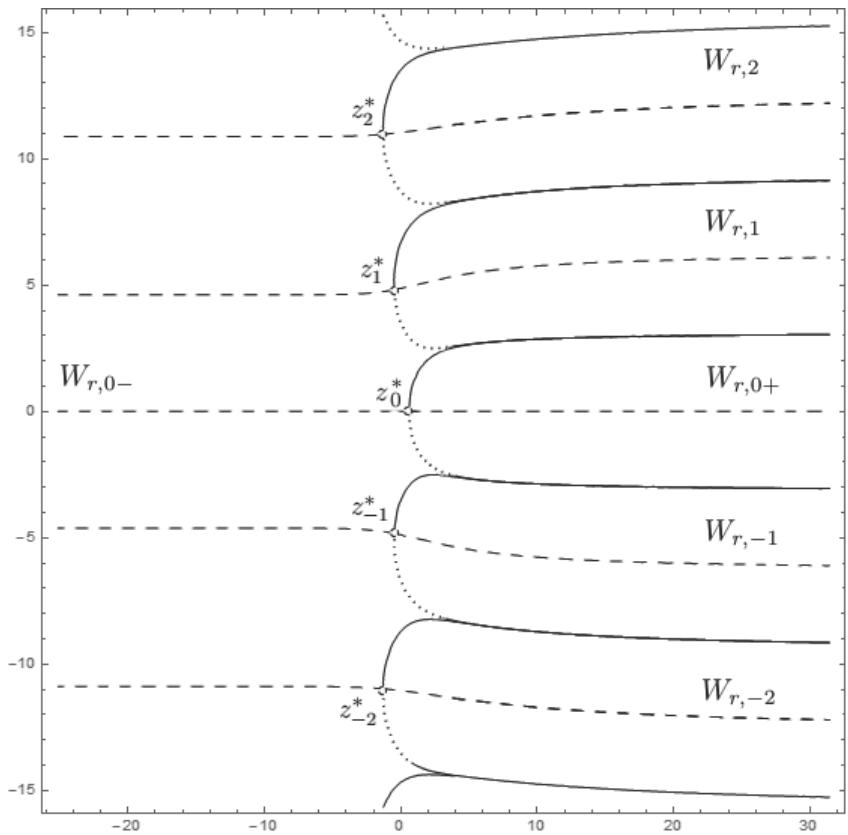


FIGURE 42
The image sets of the branches of the r-Lambert function when $r < 0$.

Now we determine the other two curves, $\Gamma^{(1)}_{0,\pm}$. These are the images of the branch cut (6.30). By CCC, we make the line $L_{0-,0+}$ a part of $W_{r,0-}$, so the curve $\Gamma^{(1)}_{0,-}$, which is the dotted line going from z^*_0 downward to the right in Figure 42, belongs to the image of $W_{r,0-}$. And $\Gamma^{(1)}_{0,+}$, which is the solid line starting from z^*_0 going upward to the right, is part of the image set of $W_{r,0+}$.

We note that for $r < 0$ the sign of b^*_k is negative when $k > 0$ (and $b^*_{-k} = -b^*_k$). This tells us that the $W_{r,0-}$ branch maps the lower half-plane into the upper half-plane and vice versa.

Thus, for example, the set $W_{r,0-}(\{x+iy \mid x \in \mathbb{R}\})$ will be a curve below the horizontal axis in the z-plane, and it goes to plus infinity in the gap between $W_{r,0+}$ and $W_{r,1}$ for $b^*_1 < y < 0$. If $y = b^*_1 < 0$ the set $W_{r,0-}(\{x + iy \mid x \in \mathbb{R}\})$ is the dashed line going through z^*_1 and continuing on the dotted line, forming the lower boundary of $W_{r,1}$.

The other branches, including $W_{r,0+}$, map the upper half-plane into the upper half of their range while the lower half-plane is mapped to their lower half. The two halves are separated by the corresponding dashed lines in Figure 42. The horizontal lines $\{x + ib^*_k \mid x \in \mathbb{R}\}$ are mapped partially onto the corresponding dashed lines in the branch $W_{r,k}$ until z^*_k is reached, and at this point they turn *upward* forming the closure of the given branch.

In accordance with all of these, the lines

$$L_{0-,k} = \{x + iy \in \mathbb{C} : x \le a^*_k,\ y = b^*_k\} \quad (k \in \mathbb{Z}_{\setminus 0}), \tag{6.31}$$

where

$$z^*_{k+1} = W_k(-re) - 1 \quad \text{if } k > 0,$$

and

$$z^*_k = W_{k+1}(-re) - 1 \quad \text{if } k < 0$$

applied, are the branch cuts for the branches $W_{r,0-}$ and $W_{r,k}$ $(k \in \mathbb{Z}_{\setminus 0})$. See Figure 43 for the arising Riemann surface.

A closed curve at a given branch point which starts in $W_{r,0-}$ lifts up onto the Riemann surface at the sheet $W_{r,0-}$, then it eventually jumps onto the sheet of the given branch, and from this branch the curve goes back to the sheet of $W_{r,0-}$ and reaches its initial point. This tells us that the monodromy group at every branch point is C_2, the cyclic group of order two.

6.3.4 More on the points z^*_k

In this section we are going to examine the asymptotic behavior of the points z^*_k as $k \to \pm\infty$. As these determine the extremal points of the images of the branches, they have special importance.

Recall that (if we do not shift the indices)

$$\xi^*_k = \Re(W_k(-re) - 1), \tag{6.32}$$

$$\eta^*_k = \Im(W_k(-re) - 1), \tag{6.33}$$

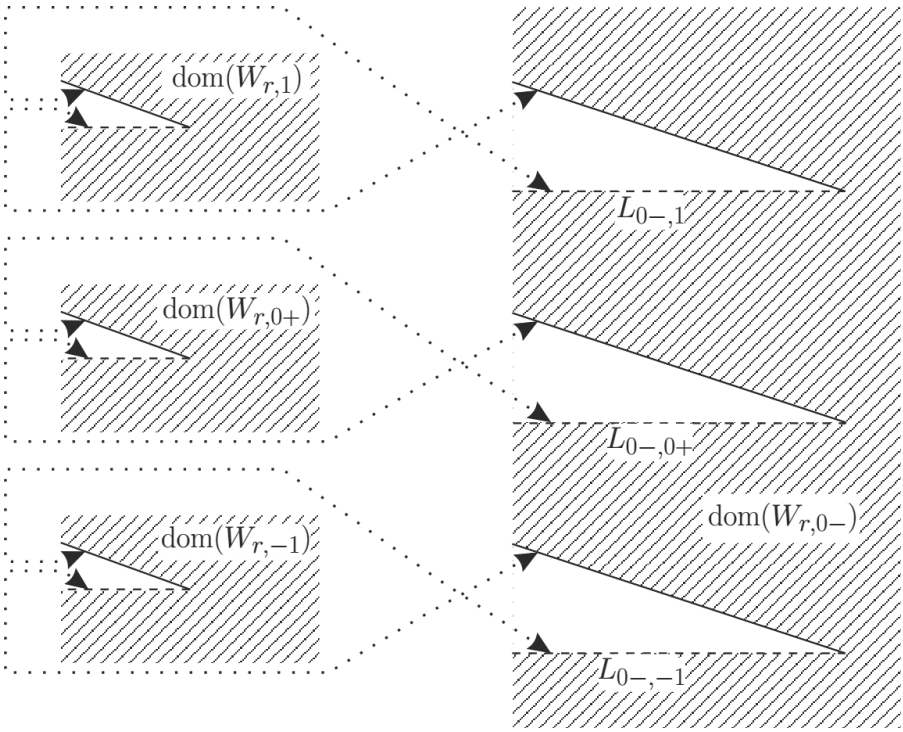


FIGURE 43
The Riemann surface of the r-Lambert function for negative r values. The lines $L_{0-,k}$ and $L_{0-,0+}$ are defined in (6.30)-(6.31).

and

$$\eta^*_{-k} = -\eta^*_k, \quad \text{and} \quad \xi^*_{-k} = \xi^*_k.$$

Thus it is enough to analyze the case when $k \to +\infty$.

The following statement is true. Let r be a non-zero real number. Then, for large k,

$$\xi^*_k \sim \log|r| - \log\left(2k\pi + \operatorname{sgn}(r)\frac{\pi}{2}\right), \tag{6.34}$$

$$\eta^*_k \sim 2k\pi + \operatorname{sgn}(r)\frac{\pi}{2}. \tag{6.35}$$

Here sgn is the signum function, and $\sim$ means that the ratio of the expressions on the two sides of $\sim$ tends to one as $k \to \infty$.

The proof is as follows. The asymptotic behavior of the Lambert function was described in (4.16). Writing it a bit differently:

$$W_k(w) = \log w + 2k\pi i - \log(\log w + 2k\pi i) + O\left(\frac{\log\log w}{\log w}\right).$$

In this expansion, the principal branch of the logarithm is taken at each occurrence.

We make use of this asymptotic expansion when the branch index k is big and w simply equals to $-re$. We then have

$$W_k(-re) - 1 \approx -1 + \log|re| + (0,\pi)i + 2k\pi i - \log(\log|re| + (0,\pi)i + 2k\pi i).$$

The big "O" term will be neglected, and the symbol $(0,\pi)$ is defined to be 0 if $-re > 0$ (i.e., if $r < 0$) and it is defined to be π if $-re < 0$ (if $r > 0$). Thus

$$\eta_k^* = \Im(W_k(-re) - 1) \approx 2k\pi + (0,\pi) - \Im(\log(\log|re| + (0,\pi)i + 2k\pi i)).$$

If k is big, $\log|re|$ and $(0,\pi)i$ can be neglected in the outer logarithm's argument, $\log|re| + (0,\pi)i + 2k\pi i$ is a complex number with angle close to $\pi/2$. Hence

$$\Im(\log(\log|re| + (0,\pi)i + 2k\pi i)) \approx \Im(\log(2k\pi i)) = \frac{\pi}{2}.$$

From here we get that

$$\eta_k^* \approx 2k\pi + (0,\pi) - \frac{\pi}{2}.$$

This, by the definition of the symbol $(0,\pi)$, is equivalent to (6.35).

The statement for ξ_k^* can be treated similarly.

We know from the former sections that the $W_{r,k}$ branches are bounded from the left if $r \neq 0$ and $k \neq 0$. Our recently proven statement says more: it states that the left extremal points of the individual branches (not being equal to $W_{r,0-}$, $W_{r,0+}$, $W_{r,\pm1}$) go to the left as k grows, and the ordinate of these points logarithmically decreases to minus infinity.

Some numerical values for ξ_k^* and η_k^*

We compare the estimations of the above theorem to the exact expressions for $z_k^* = \xi_k^* + i\eta_k^*$ given by (6.32)-(6.33).

First we choose $r = 2$. The exact value of ξ_{50} is

$$\xi_{50}^* = -5.06178,$$

while (6.34) results in the estimation

$$\xi_{50,\mathrm{estim}}^* = -5.06174.$$

For η^*_{50} we have

$$\eta^*_{50} = 315.717,$$
$$\eta^*_{50,\text{estim}} = 315.73.$$

For $r = 1/e^4$ the corresponding values for $k = 50$ are

$$\xi^*_{50} = -8.16961,$$
$$\xi^*_{50,\text{estim}} = -8.16515,$$

and

$$\eta^*_{50} = 315.702,$$
$$\eta^*_{50,\text{estim}} = 315.73.$$

For negative r, the estimations are also quite good. Taking $r = -3$,

$$\xi^*_{50} = -4.64631,$$
$$\xi^*_{50,\text{estim}} = -4.64628,$$

and

$$\eta^*_{50} = 312.577,$$
$$\eta^*_{50,\text{estim}} = 312.588.$$

For larger k the performance of these estimations is even better.

Further notes

1. **Historical remarks.** The branch structure of W_r was first published in [129], while the Riemann surface of W_r was described in [131]. (In contrast with these papers, here the w- and z-planes are interchanged, for keeping consistence with the rest of this book. Also, what is a_k and b_k there, it is a_k^* and b_k^* here.)

2. **Problems.** The theory of the r-Lambert function is more-or-less well known, as we have shown in this chapter. The main property that is still missing is the *asymptotic behavior*. It would be good to provide a formula like (4.16) for the r-Lambert function.

   It is also not known what is the radius of convergence of the series (6.12).

   Write a full algorithm to calculate W_r on arbitrary branches.

Part III

Applications

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

7

Physical applications

After building up a vast knowledge on the classical Lambert function and on its generalizations, we now start to see how to apply these functions in "real-life" problems, by presenting many examples from physics, biology, and other sciences. These, by their nature, are somewhat disconnected topics. This shows the variety of the fields where the Lambert function (and its generalizations) can be used.

Different areas need different background knowledge, and we cannot go back to the fundamentals for each discipline. We try, however, to be as straightforward as possible.

7.1 Falling body in resistant media

One of the simplest, but nicest, applications of the Lambert function is the description of some parameters of the motion of a body freely falling in resistant media. Before seeing this, let us consider simpler cases.

7.1.1 Free fall in vacuum

The most basic demonstration of the use of the *Newtonian equation* $F = ma$ is the description of a free body, when no force is exerted. Such a body has a constant velocity (this constant can be zero), and thus it is either in a fixed position or it moves on a straight path.

The second simplest example is the case when a constant force, say, a homogeneous and time-independent gravitational force, is exerted on the body. If the body is in vacuum, no other forces are present. Let us fix our coordinate system such that the vertical direction is along the y axis. The gravitational force points downward, which means that its value will be negative. It depends on the passive gravitational mass m of the body, so we fix the force to be $F_g = -mg$, where g is a positive constant.

By denoting the time-dependent position of the body with $y(t)$, we know that the body's velocity is

$$\dot{y}(t) = \frac{\mathrm{d}}{\mathrm{d}t} y(t),$$

DOI: 10.1201/9781003168102-7

and its acceleration is

$$\ddot{y}(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} y(t).$$

Thus, the Newtonian equation takes the form

$$m\ddot{y}(t) = F_g.$$

Here on the left, the mass m is the inertial mass of the falling body. The inertial and gravitational masses are equal (which is not a triviality in itself but a fact experimentally confirmed with high precision!), thus Newton's equation takes the form

$$\ddot{y}(t) = -g.$$

By integrating twice, we get the position of the body in an arbitrary instant of time:

$$y(t) = -\frac{g}{2}t^2 + c_1 t + c_2.$$

The integration constants c_1 and c_2 can be determined from the initial conditions. If the initial position (i.e., the position at $t = 0$) is height $h \geq 0$, and the initial velocity of the body is v_0 (which can be positive, negative or zero), then

$$y(0) = c_2 = h, \tag{7.1}$$

and

$$\dot{y}(0) = c_1 = v_0, \tag{7.2}$$

which gives rise to the final solution:

$$y(t) = -\frac{g}{2}t^2 + v_0 t + h.$$

See Figure 44 for the graphical illustration of the position as the function of time t.

Let us determine when the body reaches the ground if it was at height $h > 0$ at the beginning. If this time is t^*, we have the following equation for it:

$$y(t^*) = 0,$$

or

$$-\frac{g}{2}(t^*)^2 + v_0 t^* + h = 0.$$

This quadratic equation has two solutions:

$$t^*_{\pm} = \frac{v_0 \pm \sqrt{2gh + v_0^2}}{g}.$$

Clearly, $t^*_+ > t^*_-$, thus the physically correct solution is t^*_+ for which we simply write t^*:

$$t^* = \frac{v_0 + \sqrt{2gh + v_0^2}}{g}.$$

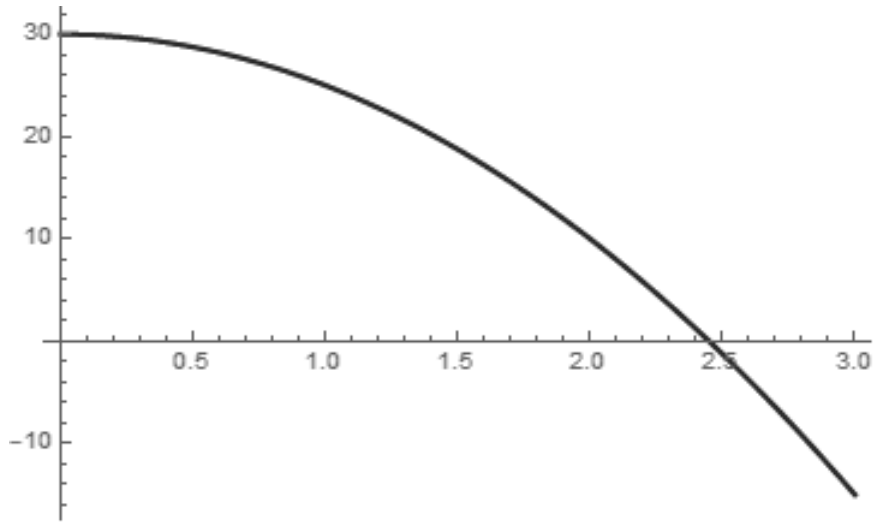


FIGURE 44
The vertical position of a body as a function of time in a homogeneous gravitational field ($v_0 = 0$, $h = 30\,m$, $g = 10\,m/s^2$).

Taking, for example,

$$g = 10\,m/s^2, \quad v_0 = 0, \quad h = 30\,m, \tag{7.3}$$

we get that

$$t^* = \sqrt{6}s \approx 2.45\,s.$$

7.1.2 Taking the resistance of the media into account

Next, let us go one step forward: if the body is not in vacuum, but it is falling in a *resistant* media, like the air, then things get a little bit complicated, and this is where Lambert's function comes into the picture. Let us see how.

The resistance exerts a force on the body which is supposed to be *proportional* to the velocity, but with opposite sign. That is, the force of resistance is of the form

$$F_r(t) = k\dot{y}(t).$$

Here k is a constant. Since the velocity $\dot{y}$ points downward (it is negative for $t < t^*$), we must have $k < 0$ so that $F_r(t) > 0$, i.e., the resistance points upwards as expected. As k times the velocity must results kilogram times acceleration, we have that

$$[k] = \frac{kg}{s}.$$

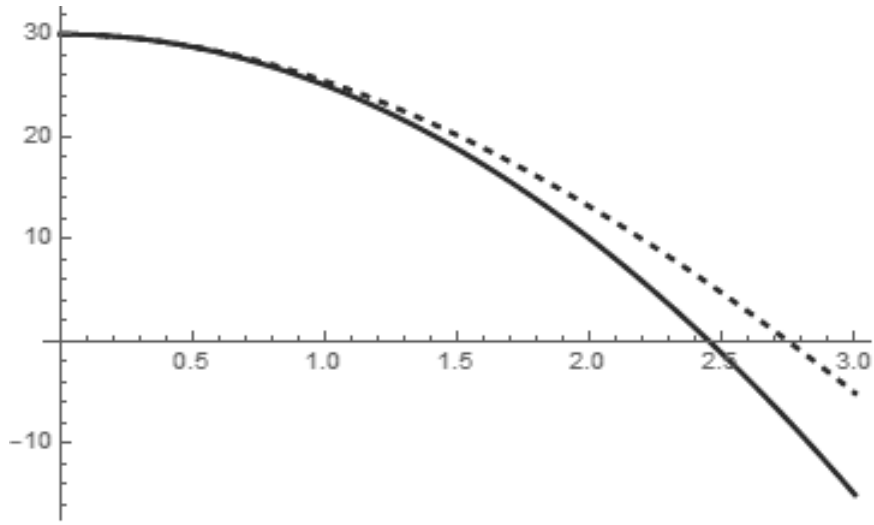


FIGURE 45
The vertical position of a body as a function of time in a homogeneous gravitational field with and without air resistance (dashed and solid curves, respectively). We set $v_0 = 0$, $h = 30\,m$, $g = 10\,m/s^2$, $k = -0.8\,kg/s$.

Thus the Newton equation becomes

$$m\ddot{y}(t) = F_g + F_r,$$

or

$$\ddot{y}(t) = -g + \frac{k}{m}\dot{y}(t). \tag{7.4}$$

The solution reads as

$$y(t) = \frac{c_1 m}{k} e^{\frac{kt}{m}} + \frac{gmt}{k} + c_2, \tag{7.5}$$

as one can check by differentiation.

If we again want (7.1)-(7.2) to hold, we get the two expressions for the integration constants c_1 and c_2:

$$c_1 = v_0 - \frac{mg}{k}, \tag{7.6}$$

$$c_2 = h - \frac{mc_1}{k}. \tag{7.7}$$

A typical situation is in Figure 45. It is nicely seen that the time after the body reaches the ground (*traveling time*) is longer than without resistance – a qualitatively obvious fact. Let us see how to determine the traveling time when air resistance is present!

Let the traveling time be t_R^*. We need to solve the equation

$$y(t_R^*) = 0,$$

where the trajectory is given by (7.5), and the constants c_1 and c_2 are in (7.6)-(7.7). This equation is of the form

$$ae^{bt} + ct + d = 0.$$

Such an equation is solvable in terms of the Lambert function, see Section 1.1.3, and the example at the end of the section. After some substitutions and algebra we get that

$$t_R^* = -\frac{c_2 k}{3g} - \frac{3}{k} W\left(\frac{c_1 k}{3g} \exp\left(-\frac{c_2 k^2}{9g}\right)\right).$$

Using the values in (7.3) plus setting $k = -0.8kg/s$, we get that

$$t_R^* \approx 2.748\, s.$$

7.2 Paramagnets and magnetization

7.2.1 The Brillouin function

The *magnetization* $\mathbf{M}$ is the distribution of magnetic moments in a region. In some cases we can consider these moments elementary if the dimensions and other characteristics of the problem allow this. Thus, elementary magnetic moment $\mathrm{d}\mathbf{m}$ can be assigned to elementary particles, molecules, or even planets if the size of the volume we are interested in is much larger than the planet itself. The magnetization then arises as the elementary magnetic moments per volume element:

$$\mathbf{M} = \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}V}.$$

Each atomic orbital (characterized by three atomic numbers) can contain two electrons so that these form a pair with opposite spins. If an electron has no such pair on its orbital, it is called *unpaired*. Unpaired electrons, having no pair to cancel out their dipole moment, behave like tiny magnets. Paramagnets are materials where unpaired electrons are present (there are exceptions, so this is not an exact definition). When an outer magnetic field $\mathbf{B}$ is applied, the spins of the unpaired electrons arrange in the direction of $\mathbf{M}$, and the material starts to behave like a magnet. Supposing that the direction of $\mathbf{B}$ is the axis z (so that $\mathbf{M}$ and $\mathbf{B}$ become scalars, M and B), the magnetization of a paramagnet is described by the equation

$$M = Ng\mu_B J B_J(x),$$

where N is the number of magnetic moments per unit volume, g is the *g-factor* (this is related to the angular momentum and spin of the particle as both contribute to the magnetic moment), μ_B is the *Bohr magneton* (a physical constant), J is the total angular momentum number (combining the spin and angular momentum) of the considered particle with the given elementary magnetic moment. Moreover,

$$x = J\frac{g\mu_B B}{k_B T},$$

where k_B is the *Boltzmann constant*, and T is the temperature of the material. Finally, the function

$$B_J(x) = \frac{2J+1}{2J}\coth\left(\frac{2J+1}{2J}x\right) - \frac{1}{2J}\coth\left(\frac{1}{2J}x\right)$$

is the *Brillouin function*.

The above considerations took the quantum mechanical nature of magnetism into account. We get the classical or macroscopic behavior of the magnetization if the total angular momentum number J tends to infinity. As it can easily be seen,

$$\lim_{J\to\infty}\frac{2J+1}{2J}\coth\left(\frac{2J+1}{2J}x\right) = \coth(x),$$

and

$$\lim_{J\to\infty}\frac{1}{2J}\coth\left(\frac{1}{2J}x\right) = \frac{1}{x},$$

thus the Brillouin function becomes

$$L(x) = \coth(x) - \frac{1}{x},$$

which is known as the *Langevin function*. For all these details, see [106].

If one is interested in the reverse problem in the classical regime, i.e., wants to know x from the magnetization, then he or she needs to resort to equations of the form

$$L(x) = \coth(x) - \frac{1}{x} = a.$$

Then $x = L^{-1}(a)$. We show that the inverse Langevin function is nothing else but a properly parametrized $(1,1)$-type generalized Lambert function (and thus the r-Lambert function).

7.2.2 The inverse Langevin function

As $\coth(x) = \frac{e^{2x}+1}{e^{2x}-1}$, the transcendental equation $L(x) = a$ can be rewritten as

$$L(x) = \frac{e^{2x}+1}{e^{2x}-1} - \frac{1}{x} = a.$$

First, let us notice that the equation is equivalent to

$$(e^{2x}+1)-\left(\frac{1}{x}+a\right)(e^{2x}-1)=0.$$

Then

$$e^{2x}\left[1-\left(\frac{1}{x}+a\right)\right]=-1+\left(\frac{1}{x}+a\right),$$

from where it easily comes that

$$e^{2x}=\frac{x(1+a)+1}{x(a-1)+1}.$$

In order to have no factor in the exponential, we substitute $y=-x/2$ in place of x so that

$$e^{-y}=\frac{-y/2(1+a)+1}{-y/2(a-1)+1}.$$

Multiply the right-hand side with -2:

$$e^{-y}=\frac{y(1+a)-2}{y(a-1)-2}.$$

Then we put the right-hand side into a more suitable form to be able to use (5.3):

$$e^{-y}=\frac{a+1}{a-1}\frac{y-\frac{2}{a+1}}{y-\frac{2}{a-1}},$$

hence

$$\frac{a-1}{a+1}=e^{y}\frac{y-\frac{2}{a+1}}{y-\frac{2}{a-1}}.$$

Its solution is then given by

$$y=W\left(\frac{\frac{2}{a+1}}{\frac{2}{a-1}};\frac{a-1}{a+1}\right).$$

Since $x=-2y$, we finally get that the inverse Langevin function is given as

$$L^{-1}(a)=-2W\left(\frac{\frac{2}{a+1}}{\frac{2}{a-1}};\frac{a-1}{a+1}\right).$$

7.3 Kepler's equation and Laplace's limit constant

7.3.1 The Kepler equation and λ

Kepler's equation

$$M=E-\varepsilon\sin(E) \tag{7.8}$$

is of fundamental importance in celestial mechanics. It expresses an explicit relation between the mean anomaly M, the eccentric anomaly E, and the eccentricity ε of the trajectory of a body subject to a central force.

It is possible to invert (7.8) in order to express E in terms of M and a power series around the eccentricity ε:

$$E = M + \sum_{n=1}^{\infty} a_n \varepsilon^n.$$

The coefficients a_n are calculated by the use of Lagrange's inversion theorem (1.51):

$$a_n = \frac{1}{2^{n-1} n!} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} (n-2k)^{n-1} \sin[(n-2k)M].$$

Surprisingly to many scientists at that time, it was discovered by Laplace that this series diverges if ε surpasses a given threshold λ, named nowadays as *Laplace limit* [58, p. 266-268]. The numerical value of λ is

$$\lambda = 0.662743419\ldots. \tag{7.9}$$

This constant is implicitly given as the *unique* positive real solution to the equation

$$\frac{\varepsilon}{1+\sqrt{1+\varepsilon^2}} \exp(\sqrt{1+\varepsilon^2}) = 1. \tag{7.10}$$

Also, λ equals $\sqrt{\mu^2 - 1}$, where μ is the unique positive solution of the transcendental equation

$$\mu = \coth(\mu).$$

S. R. Finch wrote in [58, p. 267]: "A closed-form expression for λ in terms of elementary functions is not known. An infinite series or definite integral expression for λ is likewise not known."

We now show that, if not in terms of elementary functions, λ is expressible in terms of the r-Lambert function. By using this relation, one can also give an infinite series expression for λ in terms of the Laguerre polynomials.

7.3.2 λ in terms of the $(1,1)$-Lambert function

The form of (7.10) would alert the trained eye: one might be able to solve this equation with the Lambert function; or if not, with one of its generalizations. Indeed, (7.10) can explicitly be solved in terms of the $(1,1)$-Lambert function. More precisely, we have the following result:

$$\lambda = \sqrt{\frac{1}{4} W\left(\begin{matrix} 2 \\ -2 \end{matrix}; 1\right)^2 - 1}. \tag{7.11}$$

As the $(1,1)$-type generalized Lambert function and the r-Lambert function are equivalent via (5.9), λ is readily expressible in the following way, too:

$$\lambda = \sqrt{\left(\frac{1}{2}W_{-1/e^2,0+}(4e^{-2}) + 1\right)^2 - 1}. \tag{7.12}$$

(One can see by the information given in section 6.3 that here the $W_{-1/e^2,0+}$ branch must be used, indeed.)

Finally, we remark that one can deduce another representation for the λ constant by the following infinite series:

$$\sqrt{\lambda^2+1} = 1 - 2\sum_{n=1}^{\infty} \frac{L_n'(4n)}{n} e^{-2n}. \tag{7.13}$$

Here $L_n'(4n)$ is the derivative of the nth Laguerre polynomial at the point $4n$. This last result is a direct consequence of (7.11) and (5.10) (with $t = 2$, $s = -2$, $w = 1$).

The proof of (7.11)

By the above remarks, it is enough to prove (7.11) to have everything settled.

Recall (7.10), in which we make the substitution

$$\varepsilon = \sqrt{y^2-1}.$$

Then we have

$$\frac{\sqrt{y^2-1}}{y+1} e^y = 1.$$

By taking square of both sides, and simplifying, we get

$$\frac{y-1}{y+1} e^{2y} = 1.$$

Letting $z = 2y$, we get that

$$\frac{z/2-1}{z/2+1} e^z = 1, \quad \text{or, equivalently} \quad \frac{z-2}{z+2} e^z = 1.$$

The latter equation is exactly in the form (5.3), so we have that

$$z = W\left(\begin{matrix}2\\-2\end{matrix}; 1\right).$$

Rolling back the substitutions, and remembering that the particular ε which solves the equation is just Laplace's λ, we get that

$$\lambda = \sqrt{y^2-1} = \sqrt{\left(\frac{z}{2}\right)^2 - 1} = \sqrt{\frac{1}{4}W\left(\begin{matrix}2\\-2\end{matrix}; 1\right)^2 - 1}. \tag{7.14}$$

This is (7.11), the result that we wanted to prove.

7.4 Thermodynamic properties of an ensemble in a quantum well

7.4.1 The quantum well

The main equation of classical quantum mechanics, the *Schrödinger equation*, describes the behavior of a particle (or particle system) in the presence of an outer field given by a potential V. One of the simplest potentials is the *1D infinite potential well*. A particle confined in an infinite potential well has discrete energy levels:

$$E_n = \frac{\pi^2 \hbar^2}{2md^2}, \tag{7.15}$$

where $\hbar$ is the *Planck constant*, m is the particle mass, and d is the width of the wall.

Let us denote the wave function of the particle by $\psi(x)$. The particle cannot pass through the infinite potential well, so the probability density of the particle, which is determined by $|\psi(x)|^2$, is zero outside of the wall. There are several boundary conditions which satisfy this requirement. For the moment, let us take a general surface S on which we want our wave function ψ to vanish. Then, for example,

$$\psi(s) = 0 \quad (s \in S)$$

is clearly the simplest. Such a condition is called *Dirichlet boundary condition*. Instead, we might want to set the outward normal derivative to be zero:

$$\frac{\partial \psi}{\partial n}(s) = 0 \quad (s \in S).$$

Such a condition is named after *Neumann*.

There is a third condition:

$$\frac{\partial \psi}{\partial n}(s) - \frac{1}{\Lambda}\psi(s) = 0 \quad (s \in S).$$

This was named after G. Robin (see the paper [76] which has interesting historical remarks on this naming).

7.4.2 Thermodynamics and ensembles in a quantum well

A (statistical) ensemble is a number of copies of a system, each of which represents a possible state of the system. In statistical mechanics, such an ensemble is often submerged into a heat bath of fixed temperature. One then studies the energy-exchange between the ensemble and the ambient.

The basic quantity is the *partition function*

$$Z(\beta) = \sum_n e^{-\beta E_n}.$$

Here n indexes the states of the ensemble, and

$$\beta = \frac{1}{k_B T}, \tag{7.16}$$

with k_B being the Boltzmann constant, and T is the temperature of the bath.

One can determine the main energy from the partition function as follows:

$$\langle E \rangle(\beta) = -\frac{\partial \ln Z}{\partial \beta} = \frac{\sum_n E_n e^{-\beta E_n}}{\sum_n e^{-\beta E_n}}.$$

The parameter in the Robin boundary condition alters the ground state E_0 (the smallest possible energy level). It will be

$$E_0 = -\frac{1}{\pi^2 \Lambda^2}.$$

Measuring the energies in appropriate units, the energies in (7.15) become

$$E_n = n^2 \quad (n \geq 1).$$

We shall need the third *Jacobi theta function*:

$$\theta_3(q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}. \tag{7.17}$$

The theta functions of Jacobi have connections with many branches of mathematics and physics, and their theory is beautiful. See [9, 112]. With $\theta_3(q)$ one can express the partition function:

$$Z(\Lambda; \beta) = E_0 + \sum_{n \geq 1} e^{-\beta n^2} = -\frac{1}{\pi^2 \Lambda^2} + \frac{1}{2}\left(-1 + \theta_3(e^{-\beta})\right).$$

From this, the main energy becomes

$$\langle E \rangle(\beta) = -\frac{\partial \ln Z}{\partial \beta} = -\frac{\frac{1}{\pi^2 \Lambda^2} e^{\frac{\beta}{\pi^2 \Lambda^2}} + \frac{1}{2}\frac{\mathrm{d}\theta_3(e^{-\beta})}{\mathrm{d}\beta}}{e^{\frac{\beta}{\pi^2 \Lambda^2}} + \frac{1}{2}(-1 + \theta_3(e^{-\beta}))}. \tag{7.18}$$

What is of interest from a physical point of view is that this main energy becomes *zero* at a given high temperature (that is, when β is close to zero by (7.16)). To study the behavior of the main energy at high temperatures, we apply the transformation formula

$$\theta_3(e^{-\beta}) = \left(\frac{\pi}{\beta}\right)^{1/2} \theta_3(e^{-\pi^2/\beta}).$$

From (7.17) one gets that

$$\theta_3(e^{-\pi^2/\beta}) = 1 + 2e^{-\pi^2/\beta} + 2e^{-4\pi^2/\beta} + \cdots,$$

for which one is a good approximation at extreme high temperatures. That is,

$$\theta_3(e^{-\beta}) \approx \left(\frac{\pi}{\beta}\right)^{1/2}.$$

Substituting this into the main energy expression (7.18), we get that

$$\langle E\rangle(\beta) \approx -\frac{\frac{1}{\pi^2\Lambda^2}e^{\frac{\beta}{\pi^2\Lambda^2}} - \frac{1}{4}\frac{\pi^{1/2}}{\beta^{3/2}}}{e^{\frac{\beta}{\pi^2\Lambda^2}} - \frac{1}{2} + \frac{1}{2}\frac{\pi^{1/2}}{\beta^{1/2}}}.$$

This energy indeed becomes zero when

$$\frac{1}{\pi^2\Lambda^2}e^{\frac{\beta}{\pi^2\Lambda^2}} = \frac{1}{4}\frac{\pi^{1/2}}{\beta^{3/2}}.$$

Let us solve this equation. First, we introduce the constants

$$a := \frac{1}{\pi^2\Lambda^2}, \quad \text{and} \quad b := \frac{1}{4}\pi^{1/2}.$$

Now the equation reads as

$$\beta^{3/2}e^{a\beta} = \frac{b}{a}.$$

Elevating both sides to the power 2/3:

$$\beta e^{2a\beta/3} = \left(\frac{b}{a}\right)^{2/3}.$$

Substituting

$$t = \frac{2}{3}a\beta,$$

the equation becomes

$$\frac{3}{2a}te^t = \left(\frac{b}{a}\right)^{2/3},$$

or

$$te^t = \frac{2a}{3}\left(\frac{b}{a}\right)^{2/3}.$$

Thus

$$\beta = \frac{3}{2a}t = \frac{3}{2a}W\left(\frac{2a}{3}\left(\frac{b}{a}\right)^{2/3}\right).$$

Substituting back the original values of a and b, we finally get that

$$\beta_{\langle E\rangle=0} = \frac{3\pi^2\Lambda^2}{2}W\left(\frac{1}{3\cdot 4^{1/6}}\frac{1}{\pi^{1/3}\Lambda^{2/3}}\right).$$

That is, the temperature of the heat bath at which the main energy of the ensemble becomes zero is

$$T = \frac{1}{\beta_{\langle E\rangle=0} k_B}.$$

The energy spectrum and thermodynamical properties of quantum ensembles confined in Robin potential well were studied by O. Olendski in [142, 140, 141]. The above calculations, in particular, can be found in [142], where the validity of the approximations used above was justified.

7.5 Einstein – Johnson sidewall correction in channel flow models

The *shear stress* caused by the river bank can usually be neglected when studying river's flow. But the shear stress is usually an important factor in *channel flow models*, where the size of the bank is comparable with the whole cross-section of the channel. Several *sidewall correction methods* are available in the literature; the most prominent among them is the *Einstein model* (this Einstein is not the discoverer of relativity, but its son: Hans Albert) [53, 54] and its modification by Johnson [95].

The *Prandtl's friction law* governs turbulent flows with smooth sidewall; therefore, one uses this law in flume experiments:

$$\frac{1}{\sqrt{f_w}} = 2\log_{10}\left(\mathtt{R}_w\sqrt{f_w}\right) - 0.8.$$

Here f_w is the *sidewall friction factor*, the quantity that we are interested in;

$$\mathtt{R}_w = \frac{4R_wV}{\nu}$$

is the *Reynolds number*. In this latter expression, ν is the kinematic water viscosity, R_w is the sidewall hydraulic radius, and V is the velocity.

The Einstein – Johnson correction modifies Prandtl's law to the following shape:

$$\frac{1}{\sqrt{f_w}} = 2\log_{10}\left(\frac{\mathtt{R}}{f}f_w^{3/2}\right) - 0.8. \tag{7.19}$$

Here $\mathtt{R} = 4RV/\nu$ is the overall Reynolds number, R is the overall hydraulic radius, and f is the overall friction factor.

It can be seen that (7.19) is of the form (1.14). Substituting everything, one gets that

$$f_w = \frac{1}{35}\left[\log_{10}\frac{W(x)}{x}\right]^{-2}$$

with

$$x = \left(\frac{9}{400} \frac{\mathtt{R}}{f} \right)^{\frac{1}{3}}.$$

Here we preserved the form of the solution as it was presented in [75].

7.6 Radial speed and mass loss rate of solar wind

Solar wind and stellar wind in general is the stream of charged particles that the sun and other stars eject from their atmosphere into the surrounding space.

Stellar winds affect the evolution of galaxies by injecting large amounts of matter and energy into the interstellar space. In addition, solar wind has direct impact on the earth: large bursts can cause interruption in communication, disrupt power grids, or threaten orbiting astronauts [166].

7.6.1 The governing equation of the solar wind

The stream of charged particles is traditionally treated as a hydrodynamical problem. Three fundamental equations of hydrodynamics express the conservation of mass, momentum, and energy. Respectively, these are

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{7.20}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\operatorname{div} \mathbf{u})\mathbf{u} = -\frac{1}{\rho} \nabla P + \mathbf{g}, \tag{7.21}$$

and

$$\frac{\partial \rho}{\partial t} \left(\frac{\rho u^2}{2} + \frac{3P}{2} \right) + \operatorname{div} \left[\mathbf{F}_H + \mathbf{F}_C + \rho \mathbf{u} \left(\frac{u^2}{2} + \frac{5P}{2\rho} - \frac{GM_\odot}{r} \right) \right] = -\rho^2 \Phi(T). \tag{7.22}$$

Here

$$F_C = -\kappa T^{\frac{5}{2}} \frac{\mathrm{d}T}{\mathrm{d}r}.$$

is the classical conductive flux with κ being the *Spitzer–Härm heat conductivity*. The origin of the input energy flux, $\mathbf{F}_H$, is still not entirely known; one must rely on empirical data.

To specify the above equations to the case of solar wind, one puts several conditions on the appearing functions and some additional assumptions are also made.

For example, it is known that some of the energy deposited in the corona is transported downwards to the lower atmosphere via heat conduction so the radial component of $\mathbf{F}_C$ is negative.

Also, some of the energy of the plasma particles is converted into radiation, due to collisions with bound atoms, a process that results in photon emission. This energy loss is given by the $\Phi(T)$ function. (Neither the form of Φ nor that of $\mathbf{F}_H$ and $\mathbf{F}_C$ will be needed to determine the wind speed.)

Additional realistic simplifications are made. For instance, the partial time derivative can be neglected if we are interested in time-steady solutions. The problem can be considered spherically symmetric, so it is enough to keep the radial parts of the vector differential operators. Thus, the gravitational acceleration $\mathbf{g}$ is simply

$$g = -\frac{GM_\odot}{r^2},$$

and the divergence of $\rho\mathbf{u}$ is simply

$$\operatorname{div}(\rho\mathbf{u}) = \frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2).$$

From this, and from (7.20), it already follows that

$$\rho u r^2 = \text{const.} \tag{7.23}$$

This constant is related to the total mass loss rate. From the entire sun this is

$$\dot{M} = 4\pi\rho u r^2 \tag{7.24}$$

in kg/s units.

By applying the ideal gas law, and assuming that the gas contains only one type of particle,

$$P = \frac{\rho k_B T}{m} = \rho a^2.$$

For hydrogen plasma, m is essentially the proton mass, k_B is Boltzmann's constant.

Now (7.23) and (7.21) yield the equation

$$\left(u - \frac{a^2}{u}\right)\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{2a^2}{r} - \frac{GM_\odot}{r^2}. \tag{7.25}$$

Here $M_\odot$ is the solar mass.

The above equation was deduced by E. N. Parker who also noticed that when $u(r) = a$, the term $\left(u - \frac{a^2}{u}\right)$ on the left-hand side is zero, while the right-hand-side is a non-zero constant. However, there is a value of r, denoted by r_c, where the right-hand side is zero as well. The (a, r_c) pair is called *Parker critical point*.

Integrating the left side of (7.25) from a to an arbitrary u and the right side from r_c to an arbitrary r yields an implicit transcendental equation for u and r:

$$(u^2 - a^2) - a^2 \log\left(\frac{u^2}{a^2}\right) = 4a^2 \log\left(\frac{r}{r_c}\right) + 2GM_\odot\left(\frac{1}{r} - \frac{1}{r_c}\right).$$

Introducing the variable

$$y = \left(\frac{u}{a}\right)^2,$$

our equation becomes

$$\log y - y = \log D(r), \tag{7.26}$$

where

$$D(r) = \left(\frac{r}{r_C}\right)^{-4} \exp\left[4\left(1 - \frac{r_c}{r}\right) - 1\right].$$

Equation (7.26) can be solved in terms of the Lambert function (by the help of (1.14)). Then there are two solutions for the wind speed $u(r)$, both of them describe physical processes of different nature. One is called *accelerating Parker solar wind solution*:

$$u^2(r) = \begin{cases} -a^2 W_0(-D(r)), & \text{if } r \le r_c; \\ -a^2 W_{-1}(-D(r)) & \text{if } r \ge r_c. \end{cases}$$

The other solution is the *Bondi accretion solution*:

$$u^2(r) = \begin{cases} -a^2 W_{-1}(-D(r)), & \text{if } r \le r_c; \\ -a^2 W_0(-D(r)) & \text{if } r \ge r_c. \end{cases}$$

This well illustrates the X-shaped singularity at which two solution trajectories of (u, r) intersect each other at the Parker critical point.

The above calculations were taken from [45], where comparison with the experimental data is done.

7.6.2 The mass loss rate due to solar wind

In the above calculations the mass density ρ was eliminated, thus we have no information on how much mass is lost through the outward solar wind. To determine the mass flux, we need to have information about $\mathbf{F}_H$, and $\Phi(T)$ in the first equations of the previous subsection. This is where lack of information remains. The below calculations (again from [45]) only apply to one of the several suggested mechanisms, due to R. Hammer [78].

The initial equation will be the radial part of (7.22):

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\left(F_H + F_C + \rho u\left(\frac{u^2}{2} + \frac{5P}{2\rho} - \frac{GM_\odot}{r}\right)\right)\right] = -\rho^2\Phi(T).$$

There are some realistic simplifications again so that one can get a more tractable equation. The solar atmosphere consist of two concentric layers, one is about 100 times hotter than the other. These are called *chromosphere* which is cooler, and the overlying *corona*. The transition between these is very thin compared to the radius of the sun. Thus, in place of the radial derivative we can simply write the difference between the quantities above and below the transition zone. Also, because of the thinness of this zone, we can ignore

the change in the gravitational potential and the spherical divergence. The kinetic energy term $u^2/2$ can also be neglected, because at the solar surface the wind speed is small. In the last step, we neglect the corona heating term, F_H, because the studied layers are below the volume where the most heat is deposited. What remains is the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}r}(F_C + 5nuk_BT) = -\rho^2\Phi(T).$$

Here we rewrote ρ to be the product of the particle mass m and number density n (the number of particles per unit volume) on the left-hand-side.

The balance of mass and momentum across the thin transition zone dictates that nu and nT remains approximately constant.

For the radiative loss function, an empirical expression will be used:

$$\rho^2\Phi(T) = An^2T^{-\frac{1}{2}},$$

where A is a constant. We now have

$$\frac{\mathrm{d}F_C}{\mathrm{d}r} + 5nuk_B\frac{\mathrm{d}T}{\mathrm{d}r} = -An^2T^{-\frac{1}{2}}.$$

One can rewrite this equation as

$$\xi F_C + F_C\frac{\mathrm{d}F_C}{\mathrm{d}T} = \psi, \tag{7.27}$$

with

$$\xi = 5nuk_B, \quad \text{and} \quad \psi = \kappa An^2T^2.$$

Upon integrating (7.27), we get an equation including T and F_C and values of these at the chromosphere and corona. One can neglect the values at the chromosphere as these are several magnitude smaller than the values at the coronal layer. Thus a transcendental equation arises which reads as

$$F_C + \xi T + \frac{\psi}{\xi}\log\left(1 - \frac{\xi F_C}{\psi}\right) = 0. \tag{7.28}$$

Both ξ and ψ contains the number density n at the corona. This number will be denoted by n_0. One can solve (7.28) for this quantity:

$$n_0 = \frac{5u_0k_BF_C}{\kappa AT^2(1 + W(\omega))},$$

where

$$\omega = -\exp\left(-\frac{25u_0^2k_B^2}{\kappa AT}\right) - 1.$$

The free variables are F_C, u_0, and T (all to be evaluated at the corona).

Notice that ω falls between $-1/e$ and zero, thus one can choose between two real values of the W function. This ambiguity is only virtual, because only W_{-1} belongs to a physical solution. If one took W_0, the n_0 density would be negative.

We still need to determine the mass loss rate from n_0. We recall (7.24), and that $\rho = mn$. The value of the free variables comes from empirical data and realistic approximations. After all, it comes that

$$\dot{M} \approx 9 \times 10^{-15} \sim 4 \times 10^{-14}$$

solar mass per year. This is in good agreement with the observational data which measures this loss around $2 \times 10^{-14}\ M_\odot$ per year at the solar minimum and a few times of that at the solar maximum.

7.7 Wien's displacement law

Every matter above the absolute zero temperature emits some electromagnetic radiation[1]. This radiation is easy to describe for an idealized body which absorbs all the electromagnetic radiation (in other words, it is non-reflective and opaque). Such a body is called *black body*. Black body radiation is a good approximation to describe the radiation of more realistic objects, like stars or even the human body [61].

The spectral energy density of the black body radiation is

$$\rho(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1}.$$

Here h is 2π times the Planck's constant $\hbar$, c is the speed of light in vacuum. The meaning of $\rho(\lambda, T)$ is the following: $\rho(\lambda, T)\mathrm{d}\lambda$ is the power emitted by the body at temperature T in the wavelength interval $\mathrm{d}\lambda$ per unit area. At any fixed temperature, $\rho(\lambda, T)$ has a unique peak, and as T grows, this peak shifts to the left. Moreover, at a given $\lambda_{\max}$,

$$\lambda_{\max} T = b.$$

The constant b is *Wien's displacement constant*, and this phenomenon is the *Wien displacement law*. The location of this peak can exactly be determined by the Lambert W function.

[1] Strictly speaking, this is only true for *barionic matter*.

Taking the derivative of ρ with respect to λ, and equating the result with zero, we get, after some simplification,

$$-5\exp\left(\frac{hc}{\lambda k_B T}\right) + \frac{hc}{\lambda k_B T}\exp\left(\frac{hc}{\lambda k_B T}\right) = -5.$$

Upon substituting

$$x = \frac{hc}{\lambda k_B T} = \frac{hc}{k_B b}, \tag{7.29}$$

one gets

$$(x-5)e^x = -5.$$

One obvious solution is $x = 0$, and the other (real) one is

$$x = 5 + W_0\left(-5e^{-5}\right).$$

It is now possible to express b from (7.29):

$$b = \frac{hc/k_B}{5 + W_0\left(-5e^{-5}\right)} = 2.89771 \cdot 10^{-3}\text{mK}.$$

For more connections of W with quantum statistics, see [175].

7.8 Electric field around semi-infinite parallel plates

In hydrodynamics it is a standard problem to determine the flow around a profile. Similarly, in electrodynamics one is often interested in the electromagnetic field around a charged body of given shape. The standard trick to solve this problem, at least when the problem is two-dimensional, is the following: one maps the profile or shape by a map into another one for which the problem is known or easier to solve. Once the flow/electromagnetic field is determined, the inverse of the same map can be used to transform these back to the original plane to get the solution.

We will show this method in a case where the Lambert W function plays a crucial role. The problem is the following: let two semi-infinite charged plate capacitors be given at distance d with potentials $\pm V$ in the w-plane ((ξ, η)-plane) such that their endpoints are on the η axis, see Figure 46. We want to determine the electric field in any given point of the w-plane. The third dimension can be neglected as the field is clearly translation invariant in that direction.

The idea is that we transform this problem to a more familiar one: to the case when two infinite plates are given. Let these plates be in the z-plane ((s, t)-plane). The field outside of the plates vanishes, while inside it is

$$\phi(z) = \frac{2V}{d}t,$$

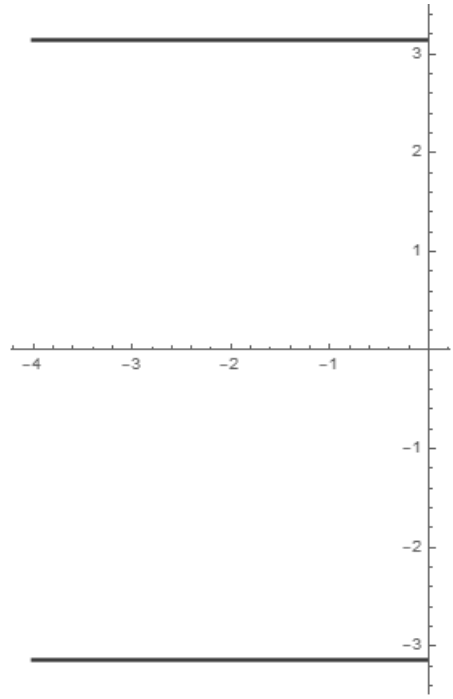


FIGURE 46
Two semi-infinite plane capacitors.

see Figure 47.

Our task is to find the transformation between the two arrangements. For simplicity, let us suppose that the lower plane is at $t = -\pi$, and the higher plane is at $t = \pi$, so the distance between the two is $d = 2\pi$, and

$$\phi(z) = \frac{V}{\pi} t. \tag{7.30}$$

One can show [177] that the transformation that maps the infinite plates into the half-plates is given by

$$T(z) = 1 + z + e^z.$$

Indeed,

$$\{\Re T(z) : z = s - \pi i,\ -\infty < s < +\infty\}) =]-\infty, 0],$$
$$\{\Im T(z) : z = s - \pi i,\ -\infty < s < +\infty\}) = -\pi,$$

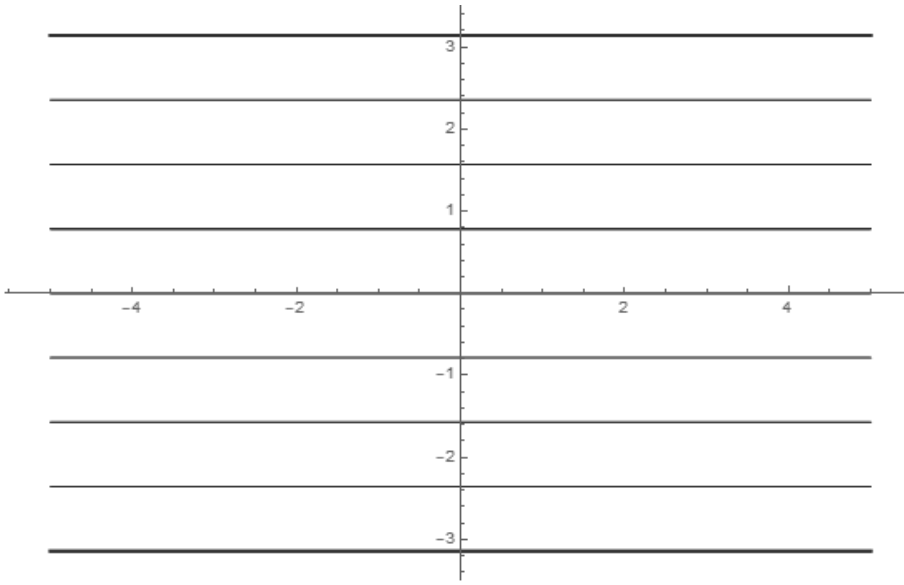


FIGURE 47
Two infinite plane capacitors in the (s,t)-plane with the equi-potential curves inside.

and

$$\{\Re T(z) : z = s + \pi i,\ -\infty < s < +\infty\}) =]-\infty, 0],$$
$$\{\Im T(z) : z = s + \pi i,\ -\infty < s < +\infty\}) = \pi.$$

One can just map the horizontal lines (equipotential curves) of the z-plane (Figure 47) by T to get the equipotential curves between the semi-infinite plates. This results in Figure 48.

We still want to know the exact value of the potential at each point in the w-plane. To do this, we need to determine the pre-image of w, and determine the potential there. The pre-image is

$$z = T^{-1}(w).$$

We thus need to solve the equation

$$1 + z + e^z = w$$

when w is fixed and z is the unknown. This equation is solvable in terms of the W function, see equation (1.12) with $b = e$. We get that

$$z = w - 1 - W_k(e^{w-1}),$$

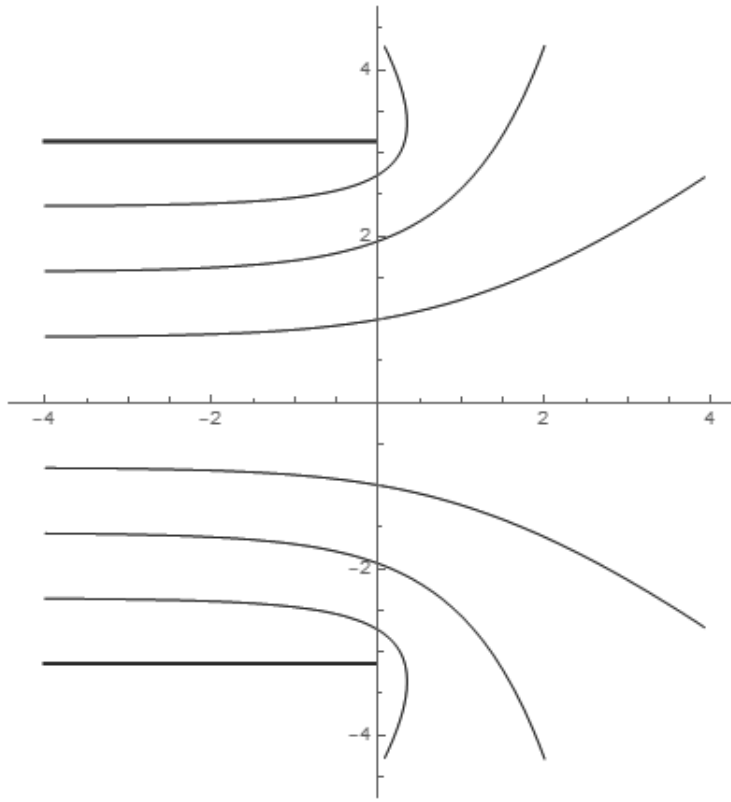


FIGURE 48
Some equipotential curves around two semi-infinite plane capacitors.

where k is an arbitrary branch index. How to choose the branch indices? The w-plane is naturally divided into horizontal strips of width 2π, the central horizontal strip is given by the capacitors. If we want that this strip be mapped by the principal branch, the strip above and below be mapped by W_1 and W_{-1}, we can use the unwinding number $\mathcal{K}$ of equation (3.6). Thus,

$$k = \mathcal{K}(z),$$

and (7.30) tells us that the potential in the z-plane is given as V/π times the imaginary part of the point in question:

$$V(w) = \frac{V}{\pi} \Im \left[w - 1 - W_{\mathcal{K}(w)}(e^{w-1}) \right].$$

These calculations were first done in [176] and were considered again in [92].

7.9 The electric potential around a collision presheath in plasma

7.9.1 Sheath and presheath

Sheaths were first described by Irving Langmuir, who wrote [111, p. 290]: "Electrons are repelled from the negative electrode while positive ions are drawn towards it. Around each negative electrode there is thus a sheath of definite thickness containing only positive ions and neutral atoms."

Another quote from K. U. Riemann [147, p. 493]: "In its simplest form the interaction of a plasma with a (more or less) absorbing wall can be characterized as follows: due to the high mobility of the electrons the wall potential will adjust itself to be negative with respect to the surrounding plasma[...]. The repulsion of electrons results in the formation of a positive space-charge region ('sheath') shielding the neutral plasma from the negative wall."

Sheath formation in plasma is of high importance in applications. There are many possible approaches [147]. The case Riemann investigated, among others, is a neutral non-magnetized plasma with one kind of singly-charged ion in contact with an absorbing wall. This wall is in the (x, y)-plane, and the plasma is limited by this wall from the right. Thus the potential is a variable of the negative z coordinate.

The physical equations that yield the shape of the potential are the following: energy conservation of the ions, continuity equation, electron density, and Poisson's equation.

The normalized kinetic energy is

$$y(z) = \frac{1}{k_B T_e} \frac{1}{2} m v(z)^2,$$

and the potential is

$$\chi(z) = -\frac{eU(z)}{k_B T_e}.$$

Here e is the absolute value of the charge of the electron.

There is a limiting velocity to which the ions need to be accelerated to enter the sheath. This limiting velocity is described by the *Bohm criterion* (originating from ion continuity and energy conservation). This can only be reached if there is a potential penetrating into the plasma region which accelerates the ions to this velocity. This region of the plasma is called *presheath*. It is customary to re-scale the problem so that

$$x = \frac{z}{L},$$

where L is large compared with the *Debye length* λ_D, and it is determined by the ion's mean free path, the ionization length and also the geometry of the system. (The Debye length is the distance characterizing the electrostatic effect around a system of charge carriers.)

There are several presheath mechanisms (see p. 497 in [147]). The one with which the Lambert function is related is the *collisional presheath*. In this case $y'(x) < \chi'(x)$, and L is taken to be the ion mean free path.

7.9.2 The potential χ in terms of W

The above-mentioned equations, applied to y and χ, now with respect to the variable x, yield a system of differential equations:

$$\frac{1}{2y}y'(x) - \chi'(x) = 0,$$
$$y'(x) - \chi'(x) = -2y.$$

(These are equations (23)-(24) specialized with (27) in [147].)

Riemann noticed that χ can be determined from these implicitly:

$$x = \frac{1}{2}\left(1 - e^{-2\chi(x)} - 2\chi(x)\right).$$

It was I. D. Dubinova [51] who realized that χ can explicitly be determined from this equation:

$$\chi(x) = \frac{1}{2}W(e^{2x-1}) - x + \frac{1}{2}.$$

The plot of this curve is given in Figure 49.

The paper of Dubinova contains more applications of W in plasma physics.

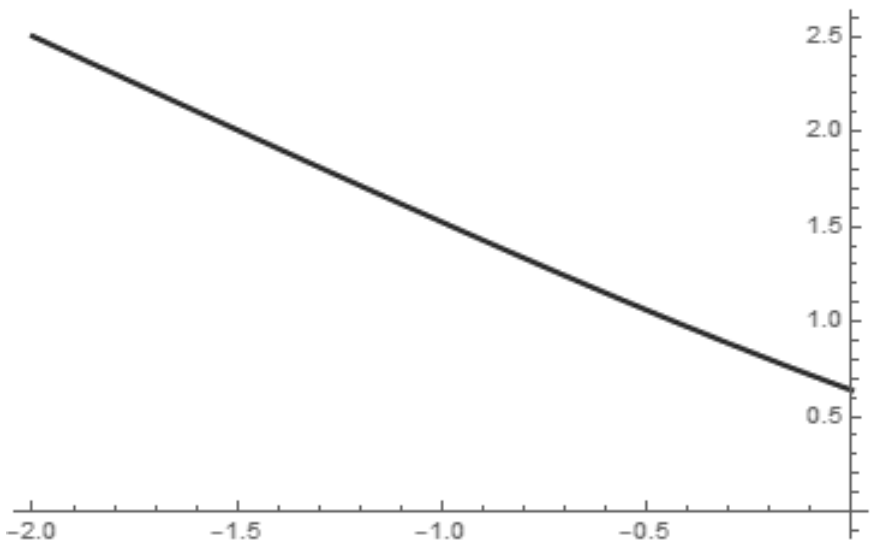


FIGURE 49
The normalized potential $\chi(x)$ on the left of an absorbing wall (presheath region).

7.10 Delta-potential model for hydrogen molecule ion

In Subsection 7.4.1 we already studied a connection of W to the Schrödinger equation. We now see another occurrence.

If the potential V is of complicated shape, the Schrödinger equation might be impossible to solve analytically. If one considers complicated molecules, this is certainly the case. Therefore, it is of importance to study simplified models which can be solved exactly and reproduce the features of the studied system, at least semi-quantitatively [63].

For the hydrogen molecule ion, Frost studied the one-dimensional potential with two Dirac delta, centered at $\pm R/2$, so that the Schrödinger equation becomes

$$-\frac{1}{2}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} - g\left[\delta\left(x-\frac{R}{2}\right)+\delta\left(x+\frac{R}{2}\right)\right]\psi(x) = E\psi(x). \tag{7.31}$$

One can find the solution by fixing the wave function ψ to be a linear combination of two (non-normalized) exponentially decaying wave:

$$\psi(x) = A\psi_a\left(-c\left|x+\frac{R}{2}\right|\right) + B\psi_b\left(-c\left|x+\frac{R}{2}\right|\right).$$

One can check that this indeed solves (7.31). One still needs to find c and the ratio of A and B. This is done by analyzing the behavior of ψ around the points $\pm R/2$. One can show (see around equation (5) in Frost's paper), that the behavior of the wave function around $x = R/2$ must be so that

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)_{R/2+\varepsilon} - \left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)_{R/2-\varepsilon} = -2g\psi\left(\frac{R}{2}\right);$$

and similarly for $-R/2$. Taking into account the form of ψ, after some algebra one gets that c must satisfy the equation

$$c = g(1 \pm e^{-cR}).$$

It is a routine now to see, that this equation is solvable with W, and

$$c_\pm = g + \frac{1}{R}W\left(\pm gRe^{-gR}\right).$$

Since $E = -c^2/2$, c_+ belongs to the lower energy level which belongs to the ground state. It is also seen that $A/B = \pm 1$, where the sign is to be chosen in correspondence with that of $c_\pm$.

The papers in which the constant c was connected to the Lambert function are due to T. C. Scott et al. [160, 161]. These authors remarked that this exact solution explains how sub-dominant terms explain the discrepancy between previous numerical solutions and the experimental results. See also [159].

Further notes

1. **Projectile motion.** The problem of the description of the motion of a body in resistant air can be generalized to plane motion, where the initial velocity vector is not vertical and where the wind speed is taken into account, too – see the paper [17] in which this problem is studied in full details. An interesting feature of this general setting is that for a critical angle the trajectory is straight. This angle is also expressible in terms of W, but it was later noticed by P. W. Andersen [8] that one does not actually need the Lambert function to determine the critical angle. See the works of Stewart [168, 170], too. The second citation contains an interesting historical part in which the pre-Newtonian ballistics is discussed. See also [143].

2. **Linear resistance.** The problem of projectile motion is not unique when a resistive force is present. The time-dependent coil current $I(t)$ of an inductance in series with a resistor is governed by the equation

$$\dot{Q}(t) + aQ(t) = b.$$

This is formally the same as (7.4) after substituting $\dot{y}(t) = I(t)$ [15, p. 513]. See also [84].

3. **Graphene nanoribbons and the offset logarithm function.** Graphene nanoribbons are an important class of next-generation carbon materials. These are narrow strips of graphene with a very high length-to-width ratio, and are classified into two types, based on their edge states: zigzag and armchair. In this setting, one problem is the description of the solutions of the Dirac equation, where periodicity properties of the wave function are dictated by the honeycomb lattice structure. This problem was first considered in [138], where the authors found that the eigenvalue condition connecting the lattice's characteristic length L, the wave number k_x and the variable z is

$$e^{-2zL} = \frac{k_x - z}{k_x + z},$$
$$e^{-2zL} = \frac{k_x + z}{k_x - z},$$

depending on the lattice point considered (see equations (76) and (78) in [138]). For the reader of this book it is quite clear that the above equations are solvable in terms of the generalized Lambert function. This was noted very recently by A. Maignan et al. [121], too, where the particular generalized Lambert function which solves the above equations was called *offset logarithm function*.

4. **The Colebrook – White equation.** This implicit equation gives the friction factor λ for fluids flowing in pipes. It reads as

$$\frac{1}{\sqrt{\lambda}} = -2\log_{10}\left(\frac{K}{3.7} + \frac{2.51}{R}\frac{1}{\sqrt{\lambda}}\right).$$

   One can solve this equation in terms of the Lambert function, see [34].

5. **Cosmology, black holes, String Theory.** An exceptionally nice appearance of W is in the Friedmann – Lemaitre – Robertson – Walker model for the geometry of the universe, see [156]. The Lambert function also has its applications in String Theory [6]. Here it is adequate to mention again that the quadratic Lambert function (see the Notes at the end of Chapter 5) is useful in general relativity.

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

8

Biology, Ecology, and Probability

8.1 Cell proliferation

8.1.1 The need for delay differential equations

Cell growths have some features in common, regardless of what kind of cells constitute the culture. Some cell growth models are described by ordinary differential equations (ODEs), but it was observed by Mitchison and Vincent [135] that the growths of several types of cell cultures show a prolonged initial step-like growth. This feature cannot be exhibited by ODEs. Therefore, it was proposed by Baker et al. [12] that delay differential equations (DDEs) be used instead of ODEs. It is demonstrated in [12] that DDEs are indeed capable of modeling cell proliferation with initial step-like growth (see figure 5.b in [12]). We are now going to see a concrete model and its relation to the (generalized) Lambert function.

8.1.2 The Baker – Bocharov – Paul – Rihan model

The proposed cell proliferation model in [12] is, in fact, a time-lag growth model, and it "provide[s] significant improvements in both the qualitative and quantitative consistency of the model with certain types of data" (p. 349). Its DDE reads as

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \rho_0 N(t) + \rho_1 N\left(t - \tau_{\mathrm{cell}}\right) + \rho_2 N'\left(t - \tau_{\mathrm{cell}}\right) \quad (t \geqq 0). \tag{8.1}$$

For the initial condition one sets

$$N(t) = \beta\Psi(t) \quad (-\tau_{\mathrm{cell}} \leq t < 0).$$

Such a DDE is called neutral (NDDE). There are five parameters in this NDDE: τ_{cell}, β, and ρ_i $(i = 1, 2, 3)$. One possible interpretation of these is given in the table below (taken from [12], p. 353.)

DOI: 10.1201/9781003168102-8

Parameter	Biological interpretation
$\tau_{\text{cell}} > 0$	The average cell-division time
$\rho_0 \geqq 0$	The proportionate rate of 'instantaneous' asynchronous cell growth
$\rho_1 \geq 0$	The proportionate rate of 'delayed' asynchronous cell growth
$0 \leqq \rho_2 \leqq 2$	A measure of the proportionate rate of 'delayed' synchronous cell growth
$0 \leqq \beta \leqq 1$	The fraction of cells dividing over the first step

In particular, if all the parameters are zero except ρ_0, the equation simplifies to

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \rho_0 N(t) \quad (t \geqq 0).$$

This is the *model for exponential growth*, since the solution reads as

$$N(t) = N \exp(\rho_0 t) \quad (t \geq 0).$$

If, instead, ρ_1 is the only non-zero parameter, then we have

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \rho_1 N\left(t - \tau_{\text{cell}}\right) \quad (t \geqq 0).$$

This is the *time-lag growth model*, since the rate of growth of the instant t depends solely on the population size τ_{cell}-time before.

There are two initial conditions studied in [12]: the uniform distribution and asynchronous growth. Here

$$\Psi(t) = N/\tau_{\text{cell}}.$$

In this case there is a steady appearance of cells over the interval $[0, \tau_{\text{cell}}[$.

The other initial condition describes synchronized growth and uses a bell-shaped distribution:

$$\Psi(t) = cE\left(\frac{2t + t_s + t_f)}{t_s - t_f}\right),$$

where c is a constant, and

$$E(t) = \begin{cases} \exp\left[-1/\left(1 - t^2\right)\right] & \text{for } |t| < 1 \\ 0 & \text{for } |t| \geqq 1. \end{cases}$$

The t_s and t_f quantities are defined so that $\tau_{\text{cell}} - t_s$ and $\tau_{\text{cell}} - t_f$ specify when the first generation of cells starts and finishes appearing, respectively $(0 \leq t_f < t_s \leq \tau_{\text{cell}})$.

Back in the time when Baker et al. described their model, the generalized Lambert function had not been defined, but it was observed by Jamilla and Mendoza [86] that the NDDE (8.1) can be solved in its full generality, i.e., for arbitrary real parameters with positive τ_{cell} in terms of the r-Lambert function. The solution $N(t)$ can be written in the form

$$N(t) = \sum_{k=-\infty}^{\infty} C_k e^{\lambda_k t},$$

where

$$\lambda_k = \rho_0 + \frac{1}{\tau_{\text{cell}}} W_{r_0,k}\left(-\rho_0 r_0 \tau_{\text{cell}} + \rho_1 \tau_{\text{cell}} e^{-\rho_0 \tau_{\text{cell}}}\right),$$

where

$$r_0 = -\rho_2 e^{-\rho_0 \tau_{\text{cell}}}.$$

The C_k constants depend on the initial condition, and k indexes the branches of the r-Lambert function.

It is worth to note, that in [86] the non-homogeneous version of (8.1) is also solved (so that the right-hand side contains an additional arbitrary function independent of N).

8.2 Random parasite and prey models

8.2.1 The Random Parasite Equation

Rogers [148, p. 369] stated that "Population models for host-parasite interactions attempt to describe the outcome of parasites search by making simple assumptions about the way in which parasites find their hosts".

Maybe the simplest of such models is due to Thompson from 1924 [173], which says that the N_{par} number of hosts parasitized is given by the expression

$$N_{\text{par}} = N(1 - e^{\text{Egg}/N}), \tag{8.2}$$

where N is the total number of hosts present, and Egg is the total number of eggs laid by the parasite. Obviously, the higher the number of eggs laid leads to higher proportion of the hosts parasitized, and the higher N is, the lower the number of the population getting affected – this is the content of (8.2).

If it is supposed that during an encounter one egg is laid, and it parasitizes the host with certainty, one can write instead

$$N_{\text{par}} = N(1 - e^{\text{Enc}/N}), \tag{8.3}$$

Holling [82] in 1959 took into account some factors the predator looks for hosts. Holling wrote that

$$\text{Enc} = a \cdot T \cdot N, \tag{8.4}$$

where a is the rate of parasite search (the amount of area covered in a unit of time T), and N is the total number of hosts, as before. Hence, the basic assumption of Holling is that the parasite searches at random over the area containing the N hosts.

If T_t is the total time during which the hosts are exposed to a parasite, and b is the time taken by the parasite to handle one host, then the correspondence between T and T_t is given by the equation

$$T = T_t - b \cdot \text{Enc}. \tag{8.5}$$

Then (8.4) and (8.5) together give

$$\text{Enc} = \frac{T_t \cdot aN}{1 + abN}. \tag{8.6}$$

This is Holling's equation 4 in [82].

If Enc is known for various N in a given experiment, then the parameters a and b can be estimated from (8.6).

Equation (8.6) also yields that the average number of parasite attacks per hosts is

$$\frac{\text{Enc}}{N} = \frac{T_t \cdot a}{1 + abN}.$$

This ratio can be substituted into (8.3), so that one gets

$$N_{\text{par}} = N\left(1 - e^{-T_t a/(1+abN)}\right). \tag{8.7}$$

This is the *Random Parasite Equation*, see Rogers [148, eq. (15)].

8.2.2 The Random Predator Equation

By using the principles of the previous subsection, one can similarly describe predator encounter; but in this case, there is a difference: the prey is eaten by the predator, so at the first (and only) encounter, the prey is removed from the population. This is taken into account by the observation that now

$$T = T_t - b \cdot N_{\text{at}}, \tag{8.8}$$

where N_{at} is the number of prey eaten by the predator. Remember that in (8.4), Enc describes the number of encounters that the predator would make with the N prey if the predator did not consume its prey. Had these encounters been distributed at random, the number of prey that would have been encountered one or more times (and therefore the total number of prey eaten) is

$$N_{\text{eat}} = N(1 - e^{-aT})$$

(which is just (8.4) and (8.3) together.) Putting (8.8) into this equation, we get

$$N_{\text{eat}} = N(1 - e^{-a(T_t - bN_{\text{at}})}). \tag{8.9}$$

If p is the density of the attacking species, we have $N_{\text{eat}} = pN_{\text{at}}$. Taking this density to be one, the final form of (8.9) is

$$N_{\text{eat}} = N(1 - e^{-a(T_t - bN_{\text{eat}})}). \tag{8.10}$$

This is the *Random Predator Equation*.

This deduction is due to Rogers [148]. Rogers's paper predates the seminal paper [39], thus he could not refer to the W function, but (8.10) can actually be solved by the Lambert W function. It comes that

$$N_{\text{eat}} = N - \frac{1}{ab} W\left(abNe^{abN-aT}\right).$$

Since N_{eat} is not greater than N, the value of W must be positive, and thus the principal branch W_0 must be taken.

The reader might find interesting the newer papers, [150] and [124], where further information can be found, and it is demonstrated how to use W in this context.

8.3 Blooming of phytoplanktons

8.3.1 Sverdrup's approach

In 1953 H. Sverdrup [171] sought to explain the spring bloom initiation of phytoplankton in the ocean. He wrote:

> In order that the vernal blooming of phytoplankton shall begin it is necessary that in the surface layer the production of organic matter by photosynthesis exceeds the destruction by respiration. [...] On certain assumptions a "critical depth" is defined. The depth of a mixed surface layer must be less than this critical depth if the phytoplankton population of the mixed layer shall increase. (p. 287)

This critical depth (below denoted by Z_c) is determined after a careful analysis of the governing equation for biomass. Sverdrup derived a transcendental equation for Z_c which was much later (in 2021) solved explicitly by Kovač et al. [180] in terms of the Lambert function. Let us see the details in the next subsection.

8.3.2 Biomass growth and critical depth

To set up the scene, we take the z axis to be pointing downward, $z = 0$ being the surface of the ocean. Let us suppose that the biomass concentration depends only on the time t and on the direction z but not on the other two

directions. Thus we denote the phytoplankton cell concentration (biomass) by $B = B(z,t)$. It is governed by a partial differential equation:

$$\frac{\partial B}{\partial t} + w\frac{\partial B}{\partial z} = \frac{1}{\chi}(P(z,t) - L(z,t)) + K_z\frac{\partial^2 B}{\partial z^2}. \tag{8.11}$$

Here K_z is the vertical eddy diffusivity, and χ is the carbon to chlorophyll ratio. Moreover, $P(z,t)$ describes photosynthesis and $L(z,t)$ represent losses in the biomass (mainly due to respiration and grazing). These two latter functions can be parametrized in numerous ways and many factors can be taken into account. For the details, see [180].

Two layers are introduced: the mixed layer at $z = Z_m$, and critical layer at $z = Z_c$. The condition that the mixed-layer depth is shallower than the critical depth (i.e., $Z_m > Z_c$) favors phytoplankton growth.

Taking into account the relevant boundary conditions, equation (8.11) can be integrated over the interval $[0, Z_m]$. For the time derivative of the cell concentration one gets

$$\frac{\partial B}{\partial t} = \frac{1}{\chi Z_m}\int_0^{Z_m}(P(z,t) - L(z,t))\mathrm{d}z.$$

It thus turns out that the direction of change in the phytoplankton biomass is determined by the overall balance between photosynthesis ($P(z,t)$) and losses in the mixed layer ($L(z,t)$). The following question arises: how deep can Z_m be for the right-hand side to remain positive? The first quantitative answer was given by Sverdrup.

To find Z_m we still need to identify P and L. Let us start with P. As photosynthesis is light-driven, and light decreases with depth in the ocean, the function $P(z,t)$ is determined by the *Beer – Lambert law*. It says that the irradiation at time t and depth z is given as

$$I(z,t) = I_0(t)e^{-Kz},$$

where $I_0(t)$ is the radiation intensity at the surface, and K is the *attenuation coefficient*. This coefficient characterizes how easily a volume of material can be penetrated by a beam of light. The larger this coefficient is, the less "transparent" the material is.

Sverdrup supposed that P depends *linearly* on I:

$$P(z,t) = B\alpha^B I(z,t).$$

Here B is the average phytoplankton biomass in the mixed layer, and α^B the initial slope of the photosynthesis-irradiance function.

For the loss rate L the assumption is made that it does not depend on the depth z, and thus its form is given as

$$L(z,t) = BL^B,$$

where L^B is the loss per unit biomass.

Sverdrup subtracts losses from production and integrates their difference over time, obtaining a function of depth $f(z)$:

$$f(z) = \int_0^D \alpha^B I_0(t) e^{-Kz} \mathrm{d}t - \int_0^{24} L^B \mathrm{d}t.$$

Here D is the daylength (time from sunrise to sunset). Time is measured in hours, as the second integral shows. Notice that irradiation is only through daytime, but loss is a ceaseless process.

Introducing the quantities

$$I_T = \int_0^D I_0(t) \mathrm{d}t,$$
$$L_T = \int_0^{24} L^B \mathrm{d}t,$$

one can integrate $f(z)$ over depth to get

$$F(Z) = \alpha^B I_T \int_0^Z e^{-Kz} \mathrm{d}z - \int_0^Z L_T^B \mathrm{d}z.$$

The last integral is simply ZL_T^B, and the first is elementary. We get that

$$F(Z) = \frac{\alpha^B I_T}{K} \left(1 - e^{-KZ}\right) - L_T^B Z.$$

The critical depth Z_c is therefore the depth where the following condition holds:

$$F(Z_c) = 0. \tag{8.12}$$

If there is to be a positive growth of the overall population, it must hold true that $Z_c > 0$ which implies that $f(0) > 0$. This latter condition, in turn, yields that

$$\frac{\alpha^B I_T}{L_T^B} > 1. \tag{8.13}$$

Let us rewrite (8.12) by using the form of F:

$$1 - e^{-KZ_c} = \frac{L_T^B}{\alpha^B I_T} K Z_c. \tag{8.14}$$

Following Kovač et al. [180], let us introduce the notations

$$\zeta_c = K Z_c,$$
$$A = \frac{\alpha^B I_T}{L_T^B}.$$

With these, (8.14) becomes

$$1 - e^{-\zeta_c} = \frac{\zeta_s}{A}.$$

This is now easily solvable by W:

$$\zeta_c = W_0\left(-Ae^{-A}\right) + A.$$

The critical depth is therefore explicitly given by the following expression:

$$Z_c = \frac{1}{K}\left(W_0\left(-Ae^{-A}\right) + A\right).$$

From the definition of A and from (8.13), it comes that the principal branch of W is to be used, and naturally, $Z_c > 0$.

The full expression for the Z_c critical depth:

$$Z_c = \frac{1}{K}\left[W_0\left(-\frac{\alpha^B I_T}{L_T^B}\exp\left(\frac{-\alpha^B I_T}{L_T^B}\right)\right) + \frac{\alpha^B I_T}{L_T^B}\right].$$

8.4 Enzyme kinetics

8.4.1 The Michaelis – Menten equation

Enzyme kinetics studies the rate of chemical reactions where enzyme is present to catalyze the process. One of the most common models for enzyme kinetics was found by L. Michaelis and M. L. Menten in 1913 [134].

Let us denote the concentration of a substrate S by $[S]$. The rate of change in the concentration of S is then simply $\mathrm{d}[S]/\mathrm{d}t$. The *Michaelis – Menten equation* states that

$$\frac{\mathrm{d}[S]}{\mathrm{d}t} = -v_{\max}\frac{[S]}{K_M + [S]}. \tag{8.15}$$

Here $v_{\max}$ is the maximum rate achieved by the system, and K_M is the substrate concentration at which the reaction rate is half of $v_{\max}$. These two constants depend on the chemical properties of the materials in question and must be measured experimentally.

8.4.2 Exact solution of the Michaelis – Menten equation

It was later observed that (8.15) can be integrated ([151, p.47-60], [163]). Indeed, consider the implicit equation

$$[S](t) + K_M \log\left(\frac{[S](t)}{[S_0]}\right) = [S_0] - v_{\max}t, \tag{8.16}$$

for $[S]$ (here $S_0 = [S](0)$ is the concentration at $t = 0$), and take the derivative:

$$[S]'(t) + K_M \frac{[S]'(t)}{[S](t)} = -v_{\max}, \quad \text{or} \quad [S]'(t)\left(1 + \frac{K_M}{[S](t)}\right) = -v_{\max}.$$

It is then clear that

$$[S]'(t) = -v_{\max} \frac{[S](t)}{K_M + [S](t)}.$$

This equation is just (8.15) with the time parameter explicitly written out.

It was further observed by S. Schnell and C. Mendoza in 1997 [158] that (8.16) can explicitly be solved if one resorts to the Lambert function. By applying (1.14), one immediately gets that

$$[S](t) = K_M W\left(\frac{[S_0]}{K_M} \exp\left(\frac{[S_0]}{K_M} - \frac{v_{\max} t}{K_M}\right)\right).$$

There are enzyme kinetics models which take into account other factors, and the Michaelis–Menten equation needs to be modified. In some cases, the Lambert function still suffices to find analytic solutions to these more elaborate models. See [69, 70] for examples.

8.4.3 Dark adaptation of the human eye

A chemical kinetics model which is very similar to the Michaelis–Menten model describes the adaptation of the human eye to darkness after exposure to bright light. The differential equation of this model is

$$\frac{\mathrm{d}O}{\mathrm{d}t} = \frac{K}{\tau} \frac{O}{K + O}.$$

Here the constants K and τ are again determined by experimentation. The method of solution is very similar to the one above, one gets that

$$O(t) = KW\left(\frac{O_0}{K} \exp\left(\frac{O_0}{K} - \frac{t}{\tau}\right)\right)$$

with $O_0 = O(0)$.

8.5 The Lindley distribution

8.5.1 The definition of the Lindley distribution

Let $F(x \mid \theta)$ be a distribution function of the random variable x, and θ be a parameter. The *fiducial distribution* [59] is then given by

$$\phi_x(\theta) = -\frac{\partial}{\partial \theta} F(x \mid \theta).$$

Clearly, one must impose several restrictions on $F(x \mid \theta)$ like differentiability and limiting values regarding the endpoint of the set for θ, and so on.

In 1958 Lindley published a paper [115] (see also [116, p. 74]) on fiducial distributions which was largely overlooked until 2008, when Ghitany et al. [68] noticed that Lindley's example for a simple fiducial distribution is sometimes better suited to describe queuing processes than the exponential distribution does, although the latter is the standard distribution for such processes. Ghitany et al. took waiting times of bank customers as a demonstrative example.

Lindley's fiducial distribution is

$$\phi_x(\theta) = \frac{\theta}{(\theta+1)^2}[1 + (1+\theta)(1+x)]xe^{-x\theta},$$

from where the *Lindley distribution function* is

$$F(x \mid \theta) = 1 - \frac{\theta + 1 + x\theta}{\theta + 1}e^{-x\theta} \quad (x, \theta > 0),$$

and the density function:

$$f(x \mid \theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-x\theta}.$$

In [68] the authors argued that the inverse transform method cannot be used to generate data from the Lindley distribution since its quantile function has no closed-form expression. In fact – as it was noted by Jodrá [93] – the quantile $QF^{-1}(x \mid \theta)$ is explicitly determinable in terms of the Lambert function.

8.5.2 Lindley and Lambert W

Let us see how to find the quantile function for the Lindley distribution. One needs to find the inverse of $F^{-1}(x \mid \theta)$, that is we need to find the solution of

$$F(x \mid \theta) = u$$

in terms of u and θ, where $0 < u < 1$. This equation is equivalent with this one:

$$(\theta + 1 + x\theta)e^{-x\theta} = (\theta+1)(1-u).$$

Multiplying both sides with $-e^{-\theta-1}$, we get

$$(-\theta - 1 - x\theta)e^{-\theta-1-x\theta} = (\theta+1)(u-1)e^{-\theta-1}.$$

Now we see that the left-hand-side is of the form ze^z, and we also know that $W(w)e^{W(w)} = w$, so

$$W\left(\frac{(\theta+1)(u-1)}{e^{\theta+1}}\right) = -(\theta+1+x\theta). \tag{8.17}$$

It is easy to check that the argument of W is between $-1/e$ and 0, and $-(\theta+1+x\theta) < -1$. This means that only the W_{-1} branch of the Lambert function can occur in (8.17). We therefore get for the quantile function that

$$Q(u \mid \theta) = -1 - \frac{1}{\theta} - \frac{1}{\theta} W_{-1}\left(\frac{(\theta+1)(u-1)}{e^{\theta+1}}\right) \quad (0 < u < 1).$$

We remark that Jodrá studied the so-called Poisson – Lindley distribution, and found that the quantile of this distribution is again expressible in terms of W (Theorem 2 in [93]).

8.6 The maximum of standard normal random variables

8.6.1 Norming constants

Let $X_1, X_2, \ldots$ be independent random variables with standard normal distribution, and set

$$Y_n = \max_{i=1,2,\ldots,n} X_i.$$

It is known, that with suitable constants a_n and b_n, the limiting distribution for the random variables

$$\frac{Y_n - b_n}{a_n}$$

is

$$\Lambda(x) = \exp(-e^{-x}),$$

the *Gumbel distribution function*. This is equivalent to saying that

$$\Phi^n(a_n x + b_n) \stackrel{n\to\infty}{\longrightarrow} \Lambda(x) \quad (-\infty < x < \infty).$$

The a_n and b_n constants are called *norming* or *normalizing constants*. Here

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \mathrm{d}t$$

is the cumulative distribution function of the *standard normal distribution*.

We now show that the most natural normalizing constants are closely related to the Lambert function.

8.6.2 The norming constants and W

It is known [60] that for any two (a_n, b_n) and (α_n, β_n) norming constant sequences

$$\lim_{n\to\infty} \frac{a_n}{\alpha_n} = 1, \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n - \beta_n}{a_n} = 0. \tag{8.18}$$

The most natural candidate for norming constants is to let b_n be determined by the equation

$$2\pi b_n^2 \exp(b_n^2) = n^2,$$

and

$$a_n = \frac{1}{b_n}.$$

It was then shown by Hall [77] that with this choice

$$\frac{C_1}{\log n} < \sup_{x\in\mathbb{R}} |\Phi^n(a_n x + b_n) - \Lambda(x)| < \frac{C_2}{\log n}.$$

Here $C_2 = 3$, but it can be taken to be 0.91 if $n \geq 10^6$. Because of (8.18), no other choice of norming constants gives better rate of convergence.

It was observed by Gasull et al. [67] that b_n is expressible in terms of W:

$$b_n = \sqrt{W\left(\frac{n^2}{2\pi}\right)}.$$

As these authors also point out, Hall's result for the estimates of b_n (and thus of a_n) immediately follows from the (4.17) asymptotics of the Lambert function.

Gasull et al. gave alternative norming constants such that with these C_2 is reduced to 1.

8.7 Superstars and retweet graphs

8.7.1 Tweets, retweets, and superstars

Bhamidi et al. [19, p. 2462] "[On *Twitter*] every user has a set of followers; these are people who have signed-up to receive the tweets of the user. Here, our focus is on retweets; these are tweets by a user who forwards a tweet that was received from another user.".

The retweets are coded into a graph in the following way: the posters of the tweets are the vertices of an undirected simple graph. There is an edge between vertices v and w if w retweets a tweet received from v, or vice-versa. This is the *retweet graph*.

Characteristically, a retweet graph contains one giant component with exceptionally large degree ("superstar vertex"), many small components and a large number of isolated vertices. The graph that was studied in details by Bhamidi et al. [19] was the retweet graph for the 2010 BET (Black Entertainment Television) Awards. In this particular retweet graph the superstar vertex has degree 992, and it is connected to more than 57% of the vertices. This superstar vertex belonged to *Lady Gaga* who was awarded in the ceremony.

The goal of Bhamidi et al. [19] was to capture the main features of the retweet graph by a model which has a minimal set of parameters. It turned out that only one parameter suffices. The construction is given in the following subsection.

8.7.2 The superstar model

The model constructed by Bhamidi et al. is a graph evolution process. The arising graphs are $G_1, G_2, \ldots$, where G_1 contains only one vertex, v_0, the *superstar*. The graph G_2 consists of the superstar v_0, and a non-superstar v_1, and an edge between the two. For $n \geq 2$, G_{n+1} is constructed by G_n by attaching the vertex v_n to either the superstar (with probability p) or to a non-superstar (with probability $q = 1 - p$). The way v_n is attached to a non-superstar is based on the *preferential attachment rule*. This means that v_n is attached to one of the non-superstars v_i $(i = 1, \ldots, n)$ with probability that is proportional to the degree of v_i in G_n.

We think about all the processes constructed on a single probability space through the obvious sequential growth mechanism so that one can make almost sure statements.

The first result of [19] describes asymptotics of the condensation phenomenon around the superstar. The result is an immediate consequence of the definition of the model and the strong law of large numbers: with probability one,

$$\lim_{n\to\infty} \frac{\deg(v_0, G_n)}{n} = p.$$

Here $\deg(v_0, G_n)$ is the degree of the superstar vertex in G_n.

The result means, in particular, that the degree of v_0 grows linearly with n. The authors then deduce the asymptotic degree distribution of G_n:

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \leq j \leq n : \deg(v_j, G_n) = k\} = v_{\mathrm{SM}}(k, p),$$

where $v_{\mathrm{SM}}(\cdot, p)$ is the probability mass function on the positive integers with values

$$v_{\mathrm{SM}}(k, p) = \frac{2-p}{1-p}(k-1)! \prod_{i=1}^{k} \left(i + \frac{2-p}{1-p}\right)^{-1}.$$

What mainly concerns us is the following question: how are these things related to our W function? Recall that G_n is a tree. View this tree as rooted at the superstar vertex v_0. Write $\mathcal{H}(G_n)$ for the graph distance of the vertex furthest from the root. This number $\mathcal{H}(G_n)$ is the height of the random tree G_n. It turns out that this height grows logarithmically with n, and

$$\lim_{n\to\infty} \frac{\mathcal{H}(G_n)}{\log n} = \frac{1-p}{2-p} \frac{1}{W\left(\frac{1}{e}\right)}.$$

A self-containing explanation would unfortunately detour us way too much, but it is quite interesting that a special value of the Lambert function appears in such a surprising context as random graphs modeling twitter events!

There is no question that W and its generalizations will appear in many more new contexts in the future.

Further notes

1. **Further applications.** There are many other applications of the W function and its generalizations, like game theory, economics, and sociology. Please see an ever-growing (but not necessarily complete) collection of such applications on the author's web page: https://sites.google.com/site/istvanmezo81/references-on-the-lambert-w-function

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

9

Mathematical applications

The Lambert function appears in mathematics sometimes surprisingly, other times its appearance is expected. In this chapter we give a handful of examples with interesting appearances of the W function. There will be examples from analysis, and combinatorics, through differential equations to Banach spaces.

9.1 The infinite power tower

9.1.1 Tetration

Let us fix a number z, and define the series

$$z_1 = z, \quad z_2 = z^{z_1}, \quad \dots \quad , z_n = z^{z_{n-1}}, \dots .$$

Clearly,

$$z_2 = z^z, \quad z_3 = z^{z^z}, \quad z_4 = z^{z^{z^z}}, \dots .$$

The operation with which z_n is constructed from z is called *tetration*; the name of z_n is the n-th tetration of z.

One question which arises – and which is related to the Lambert function – is the following: let z be fixed, and let us suppose that for this z the sequence (z_n) converges. What is its limit?

First, one needs to know when (i.e., for which z) the limit

$$z^{z^{z^{\cdot^{\cdot^{\cdot}}}}} = \lim_{n\to\infty} z_n \tag{9.1}$$

exists at all. Clearly, if $z = 1$, the series converges with limit one, but for $z = 2$ it is obviously divergent. Thus, one might expect that there is an interval around $z = 1$ for which the sequence converges. This is indeed so, as was discovered by Euler [56]. If

$$e^{-e} \le z \le e^{1/e},$$

then (9.1) exists. For further remarks on this question, see the Further Notes at the end of this chapter.

DOI: 10.1201/9781003168102-9

Numerically,

$$e^{-e} = 0.0659\ldots,$$
$$e^{\frac{1}{e}} = 1.4446\ldots.$$

Now we are going to see *what* is the limit in (9.1).

9.1.2 The power tower in terms of W

Let us write the limit of (z_n) as

$$h(z) = z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}.$$

This function clearly satisfies the equation

$$z^{h(z)} = h(z).$$

To find $h(z)$, we take logarithm of both sides:

$$h(z)\log z = \log h(z),$$

or, equivalently,

$$h(z) - \frac{1}{\log z}\log h(z) = 0.$$

This equation can be solved with the Lambert function (recall (1.14)): setting $c = h = f = 1$, $a = 0$, $b = e$, and $d = -1/\log z$, it comes that the power tower is a simple variant of the Lambert function:

$$\boxed{h(z) = -\frac{W(-\log z)}{\log z}.} \tag{9.2}$$

We can now immediately determine the values of the power tower at its two extremal points, $z = e^{-e}$, and $z = e^{1/e}$:

$$h\left(e^{-e}\right) = \frac{1}{e},$$
$$h\left(e^{\frac{1}{e}}\right) = e.$$

The graph of $h(z)$ is in Figure 50.

9.2 Counting trees

9.2.1 Networks, minimal connectedness, and labelings

Roughly speaking, a *graph* is a set of points (vertices) with possible connections (edges) between two vertices. If we think about networks, we possibly

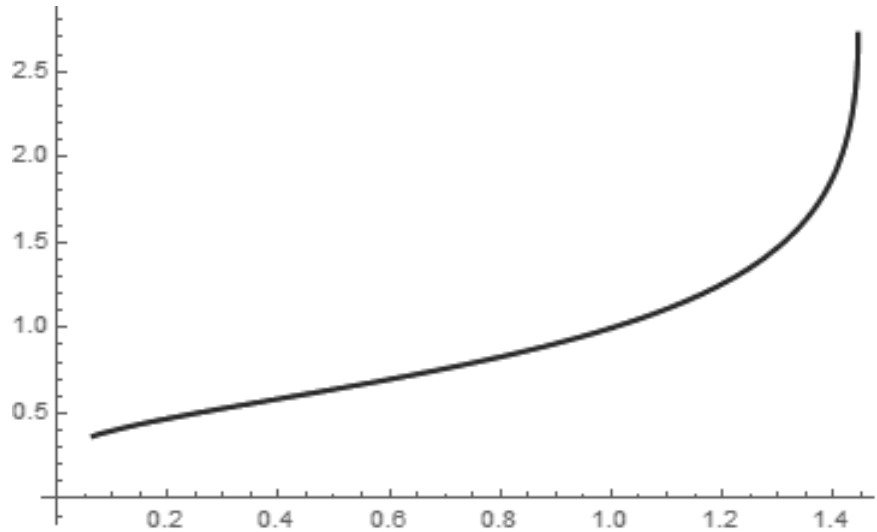


FIGURE 50
The graph of $h(z)$, where $e^{-e} \leq z \leq e^{1/e}$.

want to reach any point in the network from every other one. While this can be done in many ways, building up connections might be costly. Therefore, we can ask for *minimally* connected networks for a given set of points. Minimal connectedness means that if we remove any edge from the network, it ceases to be connected.

There are several equivalent conditions for a graph or network to be minimally connected. For example, a graph is minimally connected if and only if it contains no cycle (one cannot traverse a path of positive length from a given point such that the path ends in the same point). Also, a graph of n vertices is minimally connected if and only if it contains $n - 1$ edges. For the proofs, see Bóna's book [20, section 5.1].

A minimally connected graph is also called a *tree*. It is often useful in applications to *label* the vertices of trees, all vertices having different labels. If the tree contains n vertices, the labels can come from the set $\{1, 2, \ldots, n\}$. A labeled tree on five vertices is seen in Figure 51.

The question then arises: how many labeled trees are there on n vertices? The answer is given by a theorem attributed to Cayley [30] (although it was C. W. Borchardt [21] who discovered it).

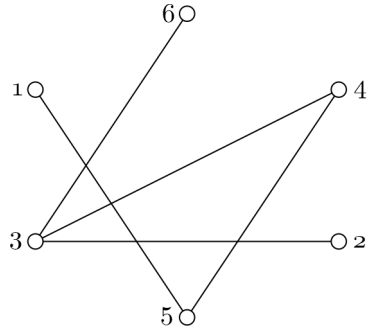


FIGURE 51
A labeled tree of 5 vertices.

9.2.2 Cayley's theorem, generating functions, and W

Cayley's theorem states that there are n^{n-2} labeled trees on n vertices (where $n \geq 1$). See [20, theorem 5.10] for a proof.

This is all nice, but how is this related to our Lambert function? In combinatorics, the main question is enumerating a set of objects. The answer to this question is often a sequence of (integer) numbers, as it is well seen in the case of the Cayley theorem: the set of labeled trees on n vertices has n^{n-2} elements. Once an integer sequence, say (a_n) is given, we are interested in its properties: how rapidly it increases (if it does), what are its divisibility properties, and so on. The *generating function* is an indispensable tool in the toolkit of combinatorialists to answer the above (and many other) questions (see an example in the next section).

The generating function of the sequence (a_n) is the series

$$f(x) = \sum_{n=1}^{\infty} a_n x^n.$$

Sometimes (a_n) grows so rapidly that such a series is convergent only at $x = 0$. To "pull back" the sequence so that we have a larger domain of convergence, we are often better to use the *exponential generating function* of the sequence, which is

$$F(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{n!}.$$

(The sum often starts with $n = 0$, it depends whether a_0 is defined or not.)

We ask the following question: what is the exponential generating function of the integer sequence $(a_n) = (n^{n-2})$? That is, we want to determine

$$F(x) = \sum_{n=1}^{\infty} n^{n-2} \frac{x^n}{n!}. \tag{9.3}$$

Those readers who remember (1.52) may suspect that $F(x)$ here has something to do with $W(x)$; and this is indeed so. Let us try to transform (9.3) so that we can express it by the W function. We start with the observation that $F'(x) = -W(-x)/x$:

$$xF'(x) = x\left(\sum_{n=1}^{\infty} n^{n-2}\frac{x^n}{n!}\right)' = \sum_{n=1}^{\infty} n^{n-2}\frac{x^n}{(n-1)!} =$$

$$\sum_{n=1}^{\infty} n\cdot n^{n-2}\frac{x^n}{n!} = \sum_{n=1}^{\infty} n^{n-1}\frac{x^n}{n!} = -W(-x),$$

where the last step comes by applying (1.52). Thus, indeed,

$$F'(x) = -\frac{W(-x)}{x}.$$

Hence

$$F(x) = -\int \frac{W(-x)}{x}\mathrm{d}x.$$

We have seen integrals involving the W function in Subsection (1.3.6). The trick – as it is usual when integrating inverse functions – is that we set $W(-x) = u$, so $-x = ue^u$ by inversion. From this, it comes that $\mathrm{d}x = -e^u(1+u)\mathrm{d}u$. This is all we need to evaluate the above integral:

$$F(x) = -\int \frac{W(-x)}{x}\mathrm{d}x = -\int \frac{u}{-ue^u}(-e^u(1+u))\mathrm{d}u = -\int (1+u)\mathrm{d}u =$$

$$-u - \frac{1}{2}u^2 + c = -W(-x) - \frac{1}{2}W^2(-x) + c.$$

We still need to determine the constant c. Since $F(0) = 0$ (the series starts with x), and $W(0) = 0$ as well, it comes that $c = 0$, whence our final result follows: the exponential generating function of the sequence enumerating the labeled trees is

$$F(x) = -W(-x) - \frac{1}{2}W^2(-x).$$

Quite nice, isn't it?

A recent paper on the tree and related functions is [98]. (The T_2 of that paper is our $F(x)$.)

9.3 The asymptotics of the Bell numbers

9.3.1 Number of groupings – The Bell numbers

Now, that we are studying combinatorial connections of the W function, let us see another, quite different appearance of the Lambert function in this area.

When we introduced the generating functions in the last section, we said that they are useful in the description of asymptotic behavior of sequences. We are now going to show a prominent example.

Let us fix an n element set and study the following problem. We want to put the elements into some non-empty disjoint groups. The question is, how many ways are there to perform this task? This number, for a fixed n, is denoted by B_n and called *Bell number*.

For $n = 2$, we have two choices: the elements are in one group together, or they separately form a group on their own. Thus $B_2 = 2$ (it is even easier to see that $B_1 = 1$).

If $n = 3$, the following possibilities are there:

$$1, 2, 3 \qquad 1 \mid 2, 3 \qquad 2 \mid 1, 3 \qquad 3 \mid 1, 2 \qquad 1 \mid 2 \mid 3.$$

Therefore $B_3 = 5$.

It is not hard to determine the exponential generating function of the Bell numbers (see [130, p. 45]):

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = e^{e^x - 1}. \tag{9.4}$$

Here we will be interested in the asymptotic behavior of the sequence (B_n). To do this analysis, we shall need a method from analytic combinatorics which will be introduced in the next section. Some preliminary knowledge on complex function theory (especially on entire functions) will be necessary.

9.3.2 The asymptotics of the Bell numbers with the aid of W

First here is a rough heuristic argument that later we make more precise: the growth of an *entire* function f is related to the growth of the coefficients (a_n) – this is a well-known observation from complex function theory [114, 113]. The growth of an entire function is measured by the modulus of growth:

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

(Maximum can be written in place of sup, since the supremum is attained, and the modulus of growth is an increasing function by the *maximum modulus principle* – see [35, Chapter VI].)

Hayman proved [80] that if f is an admissible entire function, then $M_f(r) = f(r)$, and two auxiliary functions help to find the asymptotics of a_n. These are

$$a(r) = r\frac{f'(r)}{f(r)},$$

and

$$b(r) = ra'(r).$$

(What admissibility is will not be a concern for us, it is enough to know that the function in (9.4) is an entire admissible function. For more details, see Hayman's paper or the book of Wilf [184].)

Let r_n be the positive real root of the equation $a(r) = n$ (admissibility yields that such a unique r_n indeed exists). Then, if $f(z)$ is the generating function of the sequence a_n, then

$$a_n \sim \frac{f(r_n)}{r_n^n \sqrt{2\pi b(r_n)}} \quad (n \to \infty).$$

The $\sim$ sign means that the ratio of the expressions on its two sides tends to 1 as n tends to infinity.

Now taking $f(r) = e^{e^r - 1}$, it comes immediately that

$$a(r) = re^r,$$

and

$$b(r) = re^r(1 + r).$$

Hence r_n is nothing else but $W(n)$! We have concluded that

$$\frac{B_n}{n!} \sim \frac{1}{\sqrt{2\pi}} \frac{e^{e^{W(n)} - 1}}{W(n)^n \sqrt{W(n)e^{W(n)}(1 + W(n))}}.$$

We can make this a bit simpler: we take the e^{-1} constant to the front of the expression and notice that $W(n)e^{W(n)}$ is just n:

$$\frac{B_n}{n!} \sim \frac{1}{e\sqrt{2\pi}} \frac{e^{e^{W(n)}}}{W(n)^n \sqrt{n(1 + W(n))}}.$$

Asymptotically, we do not alter much the expression if we take $W(n)$ in place of $1 + W(n)$, since

$$1 + W(n) \sim W(n).$$

Thus

$$\frac{B_n}{n!} \sim \frac{1}{e\sqrt{2\pi}} \frac{1}{\sqrt{n}} \frac{e^{e^{W(n)}}}{W(n)^{n+1/2}}. \tag{9.5}$$

This result was mentioned in [118]. See also [47, p. 102], and [137].

9.4 The associate space of an Orlicz space

9.4.1 Basics on Orlicz spaces

Perhaps the most well-known *function space* is the $L^p(A)$ space which is the vector space of functions

$$f : A \subset \mathbb{R} \to \mathbb{C}.$$

(A is usually a finite or infinite real interval or a discrete set like $\mathbb{N}$ or $\mathbb{Z}$ – in the latter two cases, we talk about a sequence space.) This space is usually equipped with the p-norm

$$\|f\|_p = \left(\int_A |f(t)|^p \mathrm{d}t \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$. The so-defined space is a topologically complete vector space, or *Banach space*.

If $A = [0, 1]$ (or any finite interval), then it can be seen that

$$L^\infty(A) \subset L^p(A) \subset L^q(A) \subset L^1(A), \tag{9.6}$$

if $p \leq q$. Here $L^\infty(A)$ is the Banach space of functions with norm

$$\|f\|_\infty = \inf \{t > 0 : \mu(\{x \in A : |f(x)| > t\}) = 0\}.$$

Here $\mu(S)$ is the Lebesgue measure of the set S.

From now on, we fix A to be the unit interval, and we leave A from the notations. The meaning of (9.6) is that for any $p > 1$, the space L^p is contained in L^1. What is interesting is that one can "insert" additional spaces between L^p and L^1, regardless of how close p is to one. More generally, one can insert spaces between any L^p and L^q if $p \neq q$. There are two standard constructions of these quite interesting spaces. One is via the introduction of weights: they are the *Lorentz spaces*. The other is the "deformation" of the function values themselves: the so-arising spaces are the Orlicz spaces. See [152] for more details on these constructions.

Heuristically, if one wants to construct a norm for a set of functions which is bigger than $\|\cdot\|_p$, one needs to "push up" the values of the functions. On the other hand, we only want to push up these values "slightly" (yes, too much obscure words, but this is just a heuristic argument) so that the "polynomial part" $|f|^p$ is not disturbed. In other words, we want to include subpolynomial transformations of $|f|$. A quite simple subpolynomial function is the logarithm. Thus, naively, one would think that

$$\|f\| = \left(\int_0^1 |f(t)|^p \log |f| dt \right)^{\frac{1}{p}} \tag{9.7}$$

is a possibility. Unfortunately, for small values of f the logarithm is big, so we would need to cut off small values of $|f|$. This can be done by considering

$$\|f\| = \left(\int_0^1 |f(t)|^p \log^+ |f| \mathrm{d}t \right)^{\frac{1}{p}}$$

instead of (9.7) with

$$\log^+ t = \begin{cases} 0, & \text{if } t < 1; \\ \log t, & \text{if } t \geq 1. \end{cases}$$

This modification is already almost good, but it still does not satisfy the triangle inequality. What we need to do is the following modification (the idea is that of W. A. J. Luxemburg from 1955 [119]):

$$\|f\|_\Phi = \inf \left\{ \lambda > 0 : \int_0^1 \left(\frac{|f(t)|}{\lambda} \right)^p \log^+ \left(\frac{|f(t)|}{\lambda} \right) \mathrm{d}t \leq 1 \right\}, \tag{9.8}$$

where

$$\Phi(t) = t^p \log^+ t. \tag{9.9}$$

Now the vector space

$$L_\Phi = \{ f : [0,1] \to \mathbb{C} : \|f\|_\Phi < +\infty \}$$

is a Banach space, called *Orlicz space* with *Orlicz function* Φ. One has that

$$L^p \subset L_\Phi \subset L^q \quad (p < q).$$

In this sense we meant that a space can be "inserted" between L^p and L^q. Of course, with different Orlicz functions one can insert additional spaces in between, the set of opportunities is infinite.

What is interesting for us is that the *associate space* $L'_\Phi = L_\Psi$ of L_Φ is again an Orlicz space. If Φ is given by (9.9), the Orlicz function Ψ will be expressed in terms of the Lambert W function.

9.4.2 The Orlicz space dual of L_Φ

What is the associate space for a given Banach function space X? It is known for the L^p spaces that

$$\int_0^1 |fg| \mathrm{d}\mu \leq \|f\|_p \|g\|_q,$$

where $1 \leq p \leq \infty$, and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

This is Hölder's inequality. In other words, dividing by $\|f\|_p$,

$$\int_0^1 \left| \frac{f}{\|f\|_p} g \right| \mathrm{d}\mu \leq \|g\|_q.$$

Hence, in particular, Hölder's inequality holds for all the functions f from the unit ball $B(L^p)$ of L^p. Moreover, this inequality is sharp in the sense that if we consider all the values of the integrals on the left-hand side and take the supremum, the q-norm of g is attained:

$$\|g\|_q = \sup\left\{\int_0^1 |fg|\mathrm{d}\mu : f \in B(L^p) \leq 1\right\}.$$

Thus, we infer that the L^q-norm could, in fact, be *defined* this way. Such a process for a given space X can be repeated in full generality, and one can define the norm

$$\|g\|_{X'} = \sup\left\{\int_0^1 |fg|\mathrm{d}\mu : f \in B(X) \leq 1\right\}$$

of a measurable function g. This norm is the norm associated with the norm of the space X, and the space X' of functions with finite $\|\cdot\|_{X'}$ norm is the space *associated to* X.

What is important for us is that associate spaces of Orlicz spaces can be defined, and the resulting space is again an Orlicz space. More precisely, the associate space of L_Φ is L_Ψ, where Ψ is the *conjugate* of Φ in the sense that their derivatives are generalized inverses of each other. In the case when Φ is defined by (9.9), being generalized inverse of Φ means that (1) the value of x where $\Phi'(x)$ becomes positive will be the value of $\Psi'(0)$; (2) for positive x, we have the following relation between the two functions:

$$\Phi'(x) = y \Longleftrightarrow \Psi'(y) = x. \tag{9.10}$$

More about this is in [16] and [152].

Now let us go back to our concrete Orlicz space defined by the Orlicz function (9.9).

It can easily be seen, that

$$\Phi'(t) = 0 \Longleftrightarrow t \leq e^{-\frac{1}{p}}. \tag{9.11}$$

So

$$\Psi'(0) = e^{-\frac{1}{p}}.$$

Since

$$\Phi'(x) = t^{p-1}(1 + p\log t),$$

so, as one can easily check, $\Phi'(x) = y$ exactly when

$$x = \left(\frac{pW\left(\frac{e^{(p-1)/p}(p-1)y}{p}\right)}{(p-1)y}\right)^{\frac{1}{1-p}}.$$

By (9.10) we get for the derivative of Ψ that

$$\Psi'(y) = \left(\frac{pW\left(\frac{e^{(p-1)/p}(p-1)y}{p}\right)}{(p-1)y} \right)^{\frac{1}{1-p}} \qquad (y > 0).$$

By some analysis, it can be shown that

$$\lim_{y \to 0+} \Psi'(y) = e^{-\frac{1}{p}},$$

in agreement with (9.11).

We get $\Psi(y)$ upon integrating the expression for Ψ'. It turns out that

$$\Psi(y) = \left(\frac{1}{(p-1)s} \tilde{\Psi}(y) \right)^{\frac{p}{1-p}} \left(\frac{1}{p} + \tilde{\Psi}(y) \right),$$

where

$$\tilde{\Psi}(y) = W\left(\frac{e^{(p-1)/p}(p-1)y}{p} \right).$$

9.5 Kapteyn series

9.5.1 The notion of Kapteyn series

The Bessel functions of the first kind

$$J_\nu(z) = \sum_{n \geq 0} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}$$

were used by Willem Kapteyn in 1893 [104] to express analytic functions $f(z)$ in the form

$$f(z) = \alpha_0 + 2 \sum_{n \geq 1} \alpha_n J_{n+\nu}((n+\nu)z).$$

Such series are now called *Kapteyn series* (or *Kapteyn – Bessel series of the first type*). The coefficients (α_n) are given by contour integrals, and the domain on which f needs to be analytic is described implicitly as follows:

$$D_a = \left\{ z \in \mathbb{C} : \Omega(z) := \left| \frac{z \exp\sqrt{1-z^2}}{1+\sqrt{1-z^2}} \right| \leq a \right\}$$

for some $a < 1$. See [14, 182] for the details.

It is known that the domain D_1 (which is, by its particular importance, often denoted by the separate letter K) is oval-like, and limited by the points

± 1 on the real line, and by $\pm i0.662743419\ldots$ on the imaginary line (see p. 270 in [182]). It is not noted in the above cited texts, but the constant 0.662743419 is, in fact, the Laplace limit constant (7.9). It is easy to see why these two constants are the same. As we saw, Laplace's limit is implicitly given by (7.10). This shows that if $\varepsilon = iz$ in (7.10), we indeed get the upper extremal point of D_1.

9.5.2 The geometry of the Kapteyn domain and the r-Lambert function

We describe the points on the boundary ∂D_a of D_a, and, in particular those which limit D_a on the real and imaginary axes. To this end, we introduce the quantities

$$p_a = \sup(D_a \cap \mathbb{R}), \quad \text{and} \quad q_a = \sup(D_a \cap (i\mathbb{R}))/i.$$

We are going to show that the numbers p_a and q_a are expressible in terms of the $(1,1)$-type generalized Lambert function and the r-Lambert function as follows: for any fixed $0 \leq a < 1$,

$$D_a = \left\{ \pm\sqrt{1 - \frac{1}{4} W\left(\begin{matrix}2\\-2\end{matrix}; -s^2 e^{2i\phi}\right)^2} : 0 \leq s \leq a,\ \phi \in]-\pi, \pi] \right\}. \tag{9.12}$$

By (5.9), we can express the generalized Lambert function with the r-Lambert function as follows:

$$W\left(\begin{matrix}2\\-2\end{matrix}; -s^2 e^{2i\phi}\right) = 2 + W_{s^2 e^{2i\phi - 2}}(-4s^2 e^{2i\phi - 2}), \tag{9.13}$$

whence, after some algebra, we get our second statement: for any fixed $0 \leq a < 1$,

$$D_a = \left\{ \pm\sqrt{-A(s,\phi) - \frac{1}{4} A(s,\phi)^2} : 0 \leq s \leq a,\ \phi \in]-\pi, \pi] \right\}, \tag{9.14}$$

where

$$A(s,\phi) = W_{s^2 e^{2i\phi - 2}}(-4s^2 e^{2i\phi - 2}). \tag{9.15}$$

It is easy to see that for the purely imaginary points in the domain D_a, it must hold true that

$$A\left(s, \frac{\pi}{2}\right) \leq 4 \quad \text{or} \quad A\left(s, \frac{\pi}{2}\right) > 0.$$

The first option is not possible because for small s the value of $A\left(s, \frac{\pi}{2}\right)$ must be small. So, $A\left(s, \frac{\pi}{2}\right) > 0$ is the only option possible, and this shows that

we must work with the principal branch of the r-Lambert function. Hence, it would be more precise to write

$$A(s,\phi) = W_{s^2e^{2i\phi-2},0+}(-4s^2e^{2i\phi-2})$$

instead of (9.15), but we can safely omit "0+", as no other branches will be involved in the present discussion.

With the knowledge given in (9.14), we can give explicit expressions for the numbers p_a and q_a: for a fixed $0 \le a < 1$ we have that

$$p_a = \sqrt{-A(a,0) - \frac{1}{4}A(a,0)^2}, \tag{9.16}$$

and

$$q_a = \frac{1}{i}\sqrt{-A\left(a,\frac{\pi}{2}\right) - \frac{1}{4}A\left(a,\frac{\pi}{2}\right)^2}. \tag{9.17}$$

More explicitly,

$$p_a = \sqrt{-W_{a^2e^{-2}}(-4a^2e^{-2}) - \frac{1}{4}W_{a^2e^{-2}}(-4a^2e^{-2})^2},$$

and

$$q_a = \frac{1}{i}\sqrt{-W_{-a^2e^{-2}}(4a^2e^{-2}) - \frac{1}{4}W_{-a^2e^{-2}}(4a^2e^{-2})^2}.$$

These results imply, in particular, that

$$p_1 = \sqrt{-W_{e^{-2}}(-4e^{-2}) - \frac{1}{4}W_{e^{-2}}(-4e^{-2})^2}.$$

The special value of this Lambert function value is easy to determine (see also at (6.27)):

$$W_{e^{-2}}(-4e^{-2}) = -2,$$

whence

$$p_1 = \sqrt{-(-2) - \frac{1}{4}(-2)^2} = 1,$$

in agreement with the observation in Watson's book [182, p. 270].

Similarly,

$$q_1 = \frac{1}{i}\sqrt{-W_{-e^{-2}}(4e^{-2}) - \frac{1}{4}W_{-e^{-2}}(4e^{-2})^2}.$$

This r-Lambert function value is also known, recall (7.12):

$$W_{-e^{-2}}(4e^{-2}) = 2(\sqrt{\lambda^2+1} - 1).$$

Hence, after some simplifications,

$$q_1 = \frac{1}{i}\sqrt{-\lambda^2} = \lambda,$$

as was observed in [182, p. 270] by numerical analysis.

9.5.3 The proofs

Since statement (9.14) directly follows from (9.12) by applying (5.9), and (9.16)-(9.17) is just a particular case of (9.14), it is enough to prove (9.12).

Let us fix the parameter a, and suppose that

$$\Omega(z) = \left| \frac{z \exp\sqrt{1-z^2}}{1+\sqrt{1-z^2}} \right| = s,$$

such that $s \leq a$. We know that such z's constitute the domain D_a. Then

$$\Omega(z) = se^{i\phi} \tag{9.18}$$

for some $\phi \in]-\pi, \pi]$. Next, we make the substitution

$$\sqrt{1-z^2} =: y \quad (\Longleftrightarrow z = \pm\sqrt{1-y^2}),$$

so that (9.18) transforms into

$$\frac{\pm\sqrt{1-y^2}e^y}{1+y} = se^{i\phi}.$$

Taking the square of both sides, and simplifying,

$$\frac{(1-y)e^{2y}}{1+y} = s^2e^{2i\phi}.$$

In the following step, we substitute

$$2y = t,$$

and get the equation

$$\frac{(1-t/2)e^t}{1+t/2} = s^2e^{2i\phi}, \quad \text{or} \quad \frac{(t-2)e^t}{t+2} = -s^2e^{2i\phi}.$$

The latter equation can now be solved directly by the $(1,1)$-type Lambert function. We have that

$$t = W\left(\begin{matrix}2\\-2\end{matrix}; -s^2e^{2i\phi}\right).$$

Rolling back to the original variable z, it holds true that any point in D_a is of the form

$$z = \pm\sqrt{1 - \frac{1}{4}W\left(\begin{matrix}2\\-2\end{matrix}; -s^2e^{2i\phi}\right)^2}$$

for some $0 \leq s \leq a$, and $\phi \in]-\pi, \pi]$, as (9.12) states.

9.5.4 The visualization of p_a, q_a and the domain D_a

Having explicit expressions for p_a, and q_a, it is now easy to visualize these functions (see Figures 52-53). We only need to find the values of

$$f(a) := W_{a^2e^{-2}}(-4a^2e^{-2}), \quad \text{and} \quad g(a) := W_{-a^2e^{-2}}(4a^2e^{-2}).$$

Based on the facts that we know for the $W_{a^2e^{-2},0+}$ branch, it can be seen that

$$f : [0,1] \to [-2,0] \quad \text{strictly decreasing,}$$
$$g : [0,1] \to [0, 2(\sqrt{\lambda^2+1}-1)] \quad \text{strictly increasing.}$$

Thus, if we start the *Halley method* to find the values of f and g, we can start the recursion from an initial point like the right endpoint of the corresponding intervals. To make this observation more apparent, we remark the following. Depending on r, there are at most two other branches of the r-Lambert function with real values, but these values are negative and on the left-hand side of the values taken by the principal branch $W_{r,0+}$.

The visualization of the domain D_a can be done as follows. We know that the rectangle with corners $\pm p_a \pm q_a i$ bound the domain D_a. The ellipse centered at the origin and with half-axes p_a and q_a is, however, a better graphical approximation of D_a. Thus, after determining p_a and q_a as it is written above, we infer that

$$\sqrt{-A(a,\phi) - \frac{1}{4}A(a,\phi)^2} \approx p_a \cos(\phi) + i q_a \sin(\phi)$$

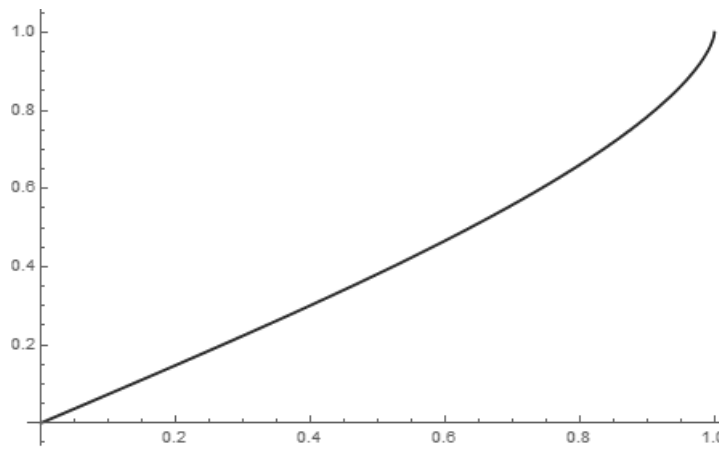


FIGURE 52
The plot of p_a for $0 \le a \le 1$.

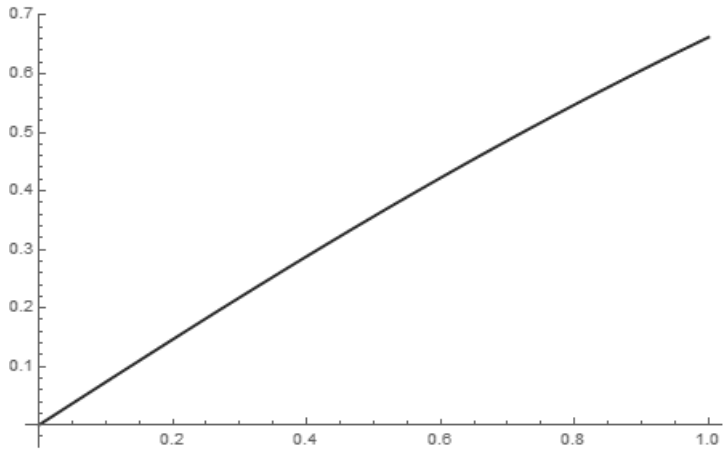


FIGURE 53
The plot of q_a for $0 \leq a \leq 1$.

for any fixed parameter a. This observation results in an approximate value for

$$A(a,\phi) = W_{a^2e^{2i\phi-2}}(-4a^2e^{2i\phi-2}).$$

This trick with the elliptical approximation is needed because the index $a^2e^{2i\phi-2}$ becomes complex, and it is not known how the r-Lambert function behaves when r is a non-real parameter. But this approximation with the ellipse gives a good start to the Halley method's iteration and we can now find the numerical value of $A(a,\phi)$ when ϕ varies. In the last step, we can determine the boundary of D_a:

$$\partial D_a = \left\{ \pm\sqrt{-A(a,\phi) - \frac{1}{4}A(a,\phi)^2} : \phi \in]-\pi,\pi] \right\}.$$

For several values of a, this boundary is depicted in Figure 54. The *Mathematica* worksheet for these figures can be downloaded from the Miscellaneous section on the author's web page[1].

[1] https://sites.google.com/site/istvanmezo81/

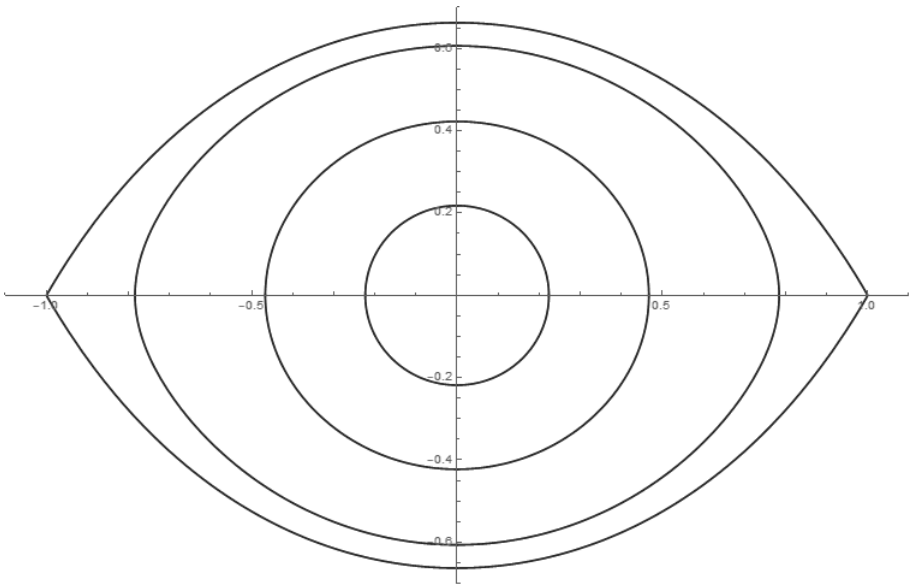


FIGURE 54
The plot of ∂D_a for $a = 1$, $a = 0.9$, $a = 0.6$, and $a = 0.3$.

9.6 The best constants for the estimate of Weierstrass canonical products

It is a classical result of Weierstrass that a complex entire function f can be expressed as an infinite product involving the zeros of f, and the *Weierstrass primary factors*

$$G(u,p) = \begin{cases} 1-u, & p=0; \\ (1-u)\exp\left[u + \frac{u^2}{2} + \cdots + \frac{u^p}{p}\right], & p>0. \end{cases}$$

in order to make the product convergent.

The study of convergence involves estimates for these primary factors. Let (a_n) be a complex sequence tending to infinity, and let $n(r)$ be the *counting function* of (a_n):

$$n(r) = |\{a_n : |a_n| \le r\}|.$$

Moreover, let us suppose that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}}$$

converges. Then the product

$$\prod_{n=1}^{\infty} G\left(\frac{z}{a_n}, p\right)$$

converges uniformly on every compact set, and satisfies the estimate

$$\log\left|\prod_{n=1}^{\infty} G\left(\frac{z}{a_n}, p\right)\right| \leq K_p r^p \left\{\int_0^r \frac{n(t)}{t^{p+1}}\mathrm{d}t + r\int_r^{\infty} \frac{n(t)}{t^{p+2}}\mathrm{d}t\right\}.$$

Here K_p is a constant (see [114, p. 29]). This is a classical result, but what is the *best* constant was only recently determined by Merzlyakov [125].

Although it seems that for a general p the expression for K_p (Theorem 4 in [125]) cannot be given in terms of standard functions, but for $p = 1$ the Lambert function gives the solution. Indeed, it is shown (see p. 239 of [125]) that

$$\log(K_1 - 1) + \frac{K_1}{2} = 0.$$

This equation is solvable by W, since it is equivalent to

$$\frac{t}{2} + \log t = -\frac{1}{2}, \quad \text{or} \quad t + 2\log t = -1,$$

where $t = k_1 - 1$. Now recall (1.14) to solve the right-hand side equation. It comes that

$$t = 2W_0\left(\frac{1}{2\sqrt{e}}\right),$$

thus

$$K_1 = 1 + 2W_0\left(\frac{1}{2\sqrt{e}}\right).$$

(K_1 is positive, so only the principal branch of W can be considered.)

The numerical value of K_1 is

$$K_1 \approx 1.477670062.$$

9.7 Delay Differential Equations

9.7.1 A simple example

If one wants to describe a system where the behavior of a characteristic attribute depends not only on the present state, but also on the earlier states of the system, then one needs to resort to *delay differential equations* (DDEs) (see [29, 165, 189] and the references therein for further information on the vast topic of DDEs). See also the examples of Chapter 8.

Perhaps the simplest among these is the DDE

$$\dot{x}(t) = \alpha x(t) + \beta x(t-\tau), \tag{9.19}$$

where the dot means differentiation with respect to t; α, β are constants, and $\tau > 0$ is the *time delay*.

The usual way of solving a DDE is using an initial guess for the form of the solution (this is the so-called *ansatz*). In this case, it is the exponential: $x(t) = e^{\lambda t}$.

Substituting this ansatz into (9.19), we get the following:

$$\lambda e^{\lambda t} = \alpha e^{\lambda t} + \beta e^{\lambda(t-\tau)}.$$

We can cancel out $e^{\lambda t}$. What remains is a transcendental equation in place of a DDE:

$$\lambda = \alpha + \beta e^{-\lambda\tau}.$$

This equation is called *characteristic equation* of the DDE, and in this case it is solvable in terms of the W function. We have done such a solution process many times, so we now just write down the result:

$$\lambda = \alpha + \frac{1}{\tau} W(\beta\tau e^{-\alpha\tau}).$$

The DDE in (9.19) is linear, thus the linear combinations of solutions are again solutions. Since λ can have infinitely many values, one for each branch of W, one infers that the function

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\lambda_k t} \tag{9.20}$$

is again a solution, where c_k constants are arbitrary complex numbers, and

$$\lambda_k = \alpha + \frac{1}{\tau} W_k(\beta\tau e^{-\alpha\tau}).$$

A solution $x(t)$ is called *stable* if

$$\lim_{t\to\infty} x(t) = 0.$$

It is clear, that (9.20) is stable, if

$$\lambda_k < 0 \quad (k \in \mathbb{Z}).$$

We said in the Further Notes of Chapter 2 that the maximal real part of $W_k(z)$ is that of $W_0(z)$, thus we get that (9.20) is stable if

$$\lambda_0 < 0,$$

that is, if

$$\alpha + \frac{1}{\tau} W_0(\beta\tau e^{-\alpha\tau}) < 0.$$

This can easily be checked for any given parameters α, β, and τ (see also Section 2.5.2 in [165], and Section 5.2 in [189]).

9.7.2 Some other examples

There are many different DDEs, and they often dictate what kind of generalizations of the classical Lambert function could be meaningful to study.

If, for instance, in (9.19) we have vector valued $x(t)$, and α, β are matrices, we are led to the study of the *matrix Lambert function* (see also the Further Notes of Chapter 5).

Considering the DDE

$$x'(t) + ax(t) + a_d x(t-\tau) + b_d x'(t-\tau) = 0$$

(which is a so-called homogeneous linear constant-coefficient neutral DDE), then the solution can be seen to be formally completely the same as in (9.20) (see [86], or check it by hand), but now the λ_k constants are given in terms of the r-Lambert function. This DDE is useful in biological applications, see Section 8.1.

Taking another example, we can consider the second order DDE

$$\ddot{x}(t) + b\dot{x}(t) + ax(t) = dx(t-\tau) + g\dot{x}(t-\tau).$$

This equation is (2) in [29] and describes a damped harmonic oscillator with non-linear feedback depending on both the state and the derivative of the state at some time in the past.

Considering again the ansatz $x(t) = e^{\lambda t}$, we get the characteristic equation

$$\lambda^2 + b\lambda + a = de^{-\lambda\tau} + g\lambda e^{-\lambda\tau}.$$

If we factor the left-hand side as

$$\lambda^2 + b\lambda + a = (\lambda - r_1)(\lambda - r_2),$$

where r_1 and r_2 are the two roots of the equation $\lambda^2 + b\lambda + a = 0$. We can rewrite the characteristic equation as

$$\frac{(\lambda - r_1)(\lambda - r_2)}{d + g\lambda} e^{\lambda\tau} = 1.$$

This equation is now solvable with the $(2, 1)$-type generalized Lambert function – compare it with (5.3).

Further notes

1. **Further analysis of the power tower.** There are many papers on the rate of convergence and other properties of the sequence (z_n) of the power tower. See [13, 55, 65, 66, 107, 136, 164, 174], and the references therein. The Wikipedia article *Tetration* and the Wolfram Mathworld page *Power tower* are also sources of historical facts, properties, notations (from which there are many), and references.

2. **The zeros of the Riemann zeta function.** The estimation of the Riemann zeta function's non-trivial zeros is a hard task. In the paper [28], the authors found a way to estimate these zeros with the aid of the Lambert W function.

3. **The asymptotics of the r-Bell numbers.** The r-Bell numbers $B_{n,r}$ generalize the Bell numbers to the case when r fixed elements must go into separate groups (see [130] and [128] for more information about this sequence). One would think that the Hayman method is applicable to the r-Bell numbers because their generating function is known and very similar to that of the Bell numbers:

$$\sum_{n=0}^{\infty} B_{n,r} \frac{x^n}{n!} = e^{e^x - 1 + rx}.$$

   Indeed, Hayman's method is applicable, but now $a(s) = se^s + rs$ (we changed the argument of a to s to avoid the clash of variables), so the r-Lambert function needs to be applied instead of the Lambert function. A formula similar to (9.5) can then be deduced. See also [36]. In fact, this connection leads the present author to define and study the r-Lambert function (first in the paper [132]), and name it this way. However, the r-Lambert function is equivalent to the $(1, 1)$-type generalized Lambert function, and the latter had been studied earlier.

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

Bibliography

[1] Formelsammlung Mathematik. *available online at https://de.wikibooks.org/wiki/Formelsammlung_Mathematik*.

[2] Inverse Symbolic Calculator. *https://isc.carma.newcastle.edu.au/index*.

[3] Mathstackexchange, question 45745. *Link to the question: https://math.stackexchange.com/questions/45745*.

[4] Oeis, the Online Encyclopedia of Integer Sequences. *http://oeis.org/*.

[5] B. V. Alekseev, A. E. Dubinov, and I. D. Dubinova. Analytical and numerical solutions of generalized dispersion equations for one-dimensional damped plasma oscillation. *High Temperature*, 43(4):479–485, 2005.

[6] C. S. Alonso. Black Holes in Supergravity with Applications to String Theory. *PhD Thesis, Universidad Autónoma de Madrid*, 2013.

[7] F. Alzahrani and A. Salem. Sharp bounds for the Lambert W function. *Integral. Transf. Spec. Funct.*, 29(12):971–978, 2018.

[8] P. W. Andersen. Comment on 'Wind-influenced projectile motion'. *Eur. J. Phys.*, 36(6):Article no. 068003, 2015.

[9] J. V. Armitage and W. F. Eberlein. *Elliptic Functions*. Cambridge University Press, 2006.

[10] H. Aslaksen. Multiple-valued complex functions and computer algebra. *SIGSAM Bulletin*, 30(2):12–20, 1995.

[11] A. Baker. *Transcendental Number Theory (2nd ed.)*. Cambridge University Press, 1990.

[12] C. T. H. Baker, G. A. Bocharov, C. A. H. Paul, and F. A. Rihan. Modelling and analysis of time-lags in some basic patterns of cell proliferation. *J. Math. Biology*, 37:341–371, 1998.

[13] I. N. Baker and P. J. Rippon. Convergence of infinite exponentials. *Ann. Acad. Sci. Fenn. Ser. A.*, 8:179–186, 1983.

DOI: 10.1201/9781003168102-9

[14] Á. Baricz, D. J. Maširević, and T. Pogány. *Series of Bessel and Kummer-Type Functions*. Springer, 2017.

[15] W. Benenson, J. W. Harris, H. Stöcker, and H. Lutz (editors). *Handbook of Physics*. Springer-Verlag, 2002.

[16] C. Bennett and R. Sharpley. *Interpolation of Operators*. Academic Press, 1988.

[17] R. C. Bernardo, J. P. Esguerra, J. D. Vallejos, and J. J. Canda. Wind-influenced projectile motion. *Eur. J. Phys.*, 36(2):Article no. 025016, 2015.

[18] B. C. Berndt. *Ramanujan's Lost Notebook – Part 1*. Springer, 1985.

[19] S. Bhamidi, J. M. Steele, and T. Zaman. Twitter event networks and the superstar model. *Ann. Appl. Probab.*, 25(5):2462–2502, 2015.

[20] M. Bóna. *Introduction to Enumerative Combinatorics*. McGraw-Hill, 2007.

[21] C. W. Borchardt. Über eine Interpolationsformel für eine Art Symmetrischer Functionen und über Deren Anwendung. *Math. Abh. der Akademie der Wissenschaften zu Berlin*, pages 1–20, 1860.

[22] J. M. Borwein and S. B. Lindstrom. Meetings with Lambert W and other special functions in optimization and analysis. *Online J. Pure Appl. Funct. Anal.*, 1(3):361–396, 2016.

[23] C. J. Bouwkamp. Solution to problem 85-16*: A conjectured definite integral. *SIAM Review*, 28(4):568–569, 1986.

[24] J. P. Boyd. Global approximations to the principal real-valued branch of the Lambert W-function. *Appl. Math. Lett.*, 11(6):27–31, 1998.

[25] R. Bradford, R. M. Corless, J. H. Davenport, and D. J. Jeffrey. Reasoning about the elementary functions of complex analysis. *Ann. Math. Artif. Intell.*, 36:303–318, 2002.

[26] M. Bronstein, R. M. Corless, J. H. Davenport, and D. J. Jeffrey. Algebraic properties of the Lambert W function from a result of Rosenlicht and of Liouville. *Integral Transf. Spec. Funct.*, 19(10):709–712, 2008.

[27] M. Bullynck. Johann Heinrich Lambert (1728-1777) – Collected Works. *available online at* http://www.kuttaka.org/~JHL/About.html.

[28] G. França and A. LeClair. Transcendental equations satisfied by the individual zeros of Riemann zeta, Dirichlet and modular L-functions. *Comm. Num. Theor. Phys.*, 9(1):1–50, 2015.

[29] S. A. Campbell. Stability and bifurcation in the harmonic oscillator with multiple, delayed feedback loops. *Dynam. Contin. Discrete Impuls. Systems*, 5:225–235, 1999.

[30] A. Cayley. A theorem on trees. *Quart. J. Pure Appl. Math.*, 23:376–378, 1889.

[31] F. Chapeau-Blondeau and A. Monir. Numerical evaluation of the Lambert W function and application to generation of generalized Gaussian noise with exponent 1/2. *IEEE Trans. Signal Process.*, 50(9):2160–2165, 2002.

[32] I. Chatzigeorgiou. Bounds on the Lambert function and their application to the outage analysis of user cooperation. *IEEE Communications Letters*, 17(8):1505–1508, 2013.

[33] J.-J. Chen and M. Lotts. Structure and randomness of the discrete Lambert map. *RHIT Undergrad. Math. J.*, 13(1):63–99, 2012.

[34] D. Clamond. Efficient resolution of the Colebrook equation. *Ind. Eng. Chem. Res.*, 48:3665–3671, 2009.

[35] J. B. Conway. *Functions of One Complex Variable (2nd ed.)*. Springer-Verlag, 1978.

[36] C. B. Corcino and R. B. Corcino. An asymptotic formula for the r-Bell numbers. *Matimyás Mat.*, 24(1):9–18, 2001.

[37] C. B. Corcino, R. B. Corcino, and I. Mező. Continued fraction expansions for the Lambert W function. *Aequationes Math.*, 93(2):485–498, 2019.

[38] R. M. Corless, H. Ding, N. J. Higham, and D. J. Jeffrey. The solution of $S * exp(S) = A$ is not always the Lambert W function of A. In *ISSAC '07*, pages 116–121.

[39] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth. On the Lambert W function. *Adv. Comput. Math.*, 5:329–359, 1996.

[40] R. M. Corless, D. Jeffrey, S. M. Watt, and J. H. Davenport. "according to Abramowitz and Stegun" or arccoth needn't be uncouth. *SIGSAM Bulletin*, 34(2):58–65, 2000.

[41] R. M. Corless and D. J. Jeffrey. The unwinding number. *SIGSAM Bulletin*, 30(2):28–35, 1996.

[42] R. M. Corless and D. J. Jeffrey. Graphing elementary Riemann surfaces. *SIGSAM Bulletin*, 32(1):11–17, 1998.

[43] R. M. Corless and D. J. Jeffrey. On the Wright ω function. In *Artificial Intelligence, Automated Reasoning, and Symbolic Computation*, pages 76–89. Springer, 2002.

[44] R. M. Corless, D. J. Jeffrey, and D. E. Knuth. A sequence of series for the Lambert W function. In *ISSAC '97*, pages 197–204.

[45] S. R. Cranmer. New views of the solar wind with the Lambert W function. *Am. J. Phys.*, 72(11):1397–1403, 2004.

[46] A. Cuyt, V. B. Petersen, B. Verdonk, H. Waadeland, and W. B. Jones. *Handbook of Continued Fractions for Special Functions*. Springer, 2008.

[47] N. G. de Bruijn. *Asymptotic Methods in Analysis (2nd ed.)*. North-Holland Publishing Company, 1961.

[48] L. Debnath and D. Bhatta. *Integral Transforms and Their Applications (2nd ed.)*. Chapman & Hall/CRC, 2016.

[49] A. Dingle and R. J. Fateman. Branch cuts in computer algebra. In *Symbolic and Algebraic Computation*, pages 250–257. ACM Press, 1994.

[50] A. E. Dubinov, I. D. Dubinova, and S. K. Saĭkov. *The Lambert W Function and its Applications to Mathematical Problems of Physics*. RFNC-VNIIEF: Sarov, Russia [in Russian]., 2006.

[51] I. D. Dubinova. Application of the Lambert W function in mathematical problems of plasma physics. *Plasma Physics Reports*, 30(10):872–877, 2004.

[52] I. D. Dubinova. Exact closed-form solutions of some nonlinear differential equations. *Differ. Equ.*, 40(8):1195–1196, 2004.

[53] H. A. Einstein. Der hydraulische oder Profilradius. *Schweizer Bauzeitung*, 103(8):89–91, 1934.

[54] H. A. Einstein. Formulas for the transportation of bedload. *Trans. ASCE*, 107:561–597, 1942.

[55] G. Eisenstein. Entwicklung von $\alpha^{\alpha^{\alpha^{\cdot^{\cdot^{\cdot}}}}}$. *J. Reine Angew. Math.*, 28:49–52, 1844.

[56] L. Euler. De formulis exponentialibus replicatis. *Leonhardi Euleri Opera Omnia, Ser. 1, 268-297. (original date of publication: 1777. Eneström index: E.489).*

[57] L. Euler. De serie Lambertina plurimisque eius insignibus proprietatibus. *Leonhardi Euleri Opera Omnia, Ser. 1, 350-369. (original date of publication: 1779. Eneström index: E.532).*

[58] S. R. Finch. *Mathematical Constants*. Cambridge University Press, 2003.

[59] R. A. Fisher. *Statistical Methods and Scientific Inference*. Oliver & Boyd, 1956.

[60] R. A. Fisher and L. H. C. Tippett. Limiting forms of the frequency distribution of the largest or smallest member of a sample. *Proc. Camb. Phil. Soc.*, 24:180–190, 1928.

[61] J. G. Flores. The human thermal system. In *Biomedical Engineering Principles*. Marcel Dekker, 1976.

[62] F. N. Fritsch, R. E. Schafer, and W. P. Crowley. Algorithm 443: Solution of the transcendental equation $we^w = x$. *Communications of the ACM*, 16:123–124, 1973.

[63] A. A. Frost. Delta-function model. I. Electronic energies of hydrogen-like atoms and diatomic molecules. *J. Chem. Phys.*, 25(6):1150–1154, 1956.

[64] T. Fukushima. Precise and fast computation of Lambert W-functions without transcendental function evaluations. *J. Comput. Appl. Math.*, 244:77–89, 2013.

[65] I. N. Galidakis. Lambert's W function and convergence of infinite exponentials in the space of quaternions. *Complex Var. Elliptic Equ.*, 51(12):1129–1152, 2006.

[66] Fu-Ch. Gao, Li-X. Han, and K. Schilling. On the rate of convergence of iterated exponentials. *J. Appl. Math. Comput.*, 39:89–96, 2012.

[67] A. Gasull, M. Jolis, and F. Utzet. On the norming constants for normal maxima. *J. Math. Anal. Appl.*, 422:376–396, 2015.

[68] M. E. Ghitany, B. Atieh, and S. Nadarajah. Lindley distribution and its application. *Math. Comput. Simulation*, 78:493–506, 2008.

[69] M. Goličnik. On the Lambert W function and its utility in biochemical kinetics. *Biochem. Engnrg. J.*, 63:116–123, 2012.

[70] Ch. T. Goudar. An explicit solution for progress curve analysis in systems characterized by endogenous substrate production. *Microbial Ecology*, 63:898–904, 2012.

[71] J. Gow. *A Short History of Greek Mathematics*. Cambridge University Press, 2010.

[72] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products (7th ed.)*. Academic Press as an imprint of Elsevier, 2007.

[73] W. B. Gragg. Matrix interpretations and applications of the continued fraction algorithm. *Rocky Mountain J. Math.*, 4:213–225, 1974.

[74] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison-Wesley, 1990.

[75] J. Guo. Exact procedure for Einstein-Johnson's sidewall correction in open channel flow. *J. Hydraul. Eng.*, 143(4):6 p., 2017.

[76] K. Gustafson and T. Abe. The third boundary condition – Was it robin's? *Math. Intell.*, 20(1):63–71, 1998.

[77] P. Hall. On the rate of convergence of normal extremes. *J. Appl. Probab.*, 16(2):433–439, 1979.

[78] R. Hammer. Energy balance of stellar coronae, I. Methods and examples. *Astrophys. J.*, 259:767–778, 1982.

[79] B. Hayes. Why W? *American Scientist*, 93:104–108, 2005.

[80] W. Hayman. A generalisation of Stirling's formula. *J. Reine Angew. Math.*, 196:67–95, 1956.

[81] N. Higham. How to program log z. *online note,* https://nhigham.com/2018/06/11/how-to-program-log-z/.

[82] C. S. Holling. Some characteristics of simple types of predation and parasitism. *Can. Ent. 91 (1959), 385-398.*, 91:385–398, 1959.

[83] A. Hoorfar and M. Hassani. Inequalities on the Lambert W function and hyperpower function. *J. Ineq. Pure. Appl. Math.*, 9(2):Article 51, 2008.

[84] A. Houari. Additional applications of the Lambert W function in physics. *Eur. J. Phys.*, 34:695–702, 2013.

[85] R. Iacono and J. P. Boyd. New approximations to the principal real-valued branch of the Lambert W-function. *Adv. Comput. Math.*, 43:1403–1436, 2017.

[86] C. Jamilla, R. Mendoza, and I. Mező. Solutions of neutral delay differential equations using a generalized Lambert W function. *Appl. Math. Comp.*, 382:Article 125334, 2020.

[87] D. J. Jeffrey. The multi-valued nature of inverse functions. *preprint, available online at* https://www.uwo.ca/apmaths/faculty/jeffrey/pdfs/inverses.pdf.

[88] D. J. Jeffrey. Not seeing the roots for the branches: multivalued functions in computer algebra. *SIGSAM Bulletin*, 38(3):57–66, 2004.

[89] D. J. Jeffrey. Branch structure and implementation of Lambert W. *Math. Comput. Sci.*, 11(3-4):341–350, 2017.

[90] D. J. Jeffrey, R. M. Corless, D. E. G. Hare, and D. E. Knuth. Sur l'inversion de $y^\alpha e^y$ au moyen de nombres de Stirling associés. *C. R. Math. Acad. Sci. Paris*, 320:1449–1452, 1995.

[91] D. J. Jeffrey and J. E. Jankowski. Branch differences and Lambert W. In *16th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing*, 2014.

[92] D. C. Jenn. Applications of the Lambert W function in electromagnetics. *IEEE Antenna and Propagation Magazine*, 44(3):139–142, 2002.

[93] P. Jodrá. Computer generation of random variables with Lindley or Poisson -– Lindley distribution via the Lambert W function. *Math. Comput. Simulation*, 81:851–859, 2010.

[94] F. Johansson. Computing the Lambert W function in arbitrary-precision complex interval arithmetic. *Numer. Algorithms*, 83:221–242, 2020.

[95] J. W. Johnson. The importance of considering sidewall friction in bed-load investigations. *Civ. Eng.*, 12(6):329–332, 1942.

[96] G. A. Jones and D. Singerman. *Complex Functions. An Algebraic and Geometric Viewpoint.* Cambridge University Press, 1987.

[97] W. B. Jones and W. J. Tron. *Continued Fractions – Analytic Theory and Applications.* Cambridge University Press, 1980.

[98] M. Josuat-Vergès. Derivatives of the tree function. *Ramanujan J.*, 38:1–15, 2015.

[99] W. Kahan. Branch cuts for complex elementary functions, or much ado about nothing's sign bit. In *the State of the Art in Numerical Analysis: Proceedings of the Joint IMA/SIAM Conference on the State of the Art in Numerical Analysis Held at the UN*, pages 165–211. Clarendon Press, 1987.

[100] G. A. Kalugin and D. J. Jeffrey. Convergence in $\mathbb{C}$ of series for the Lambert W function. *arXiv:1208.0754, 2012.*

[101] G. A. Kalugin and D. J. Jeffrey. Series transformations to improve and extend convergence. In *Lecture Notes in Computer Science, vol 6244.* Springer, 2010.

[102] G. A. Kalugin and D. J. Jeffrey. Unimodal sequences show Lambert W is Bernstein. *C. R. Math. Rep. Acad. Sci. Canada*, 33(2):50–56, 2011.

[103] G. A. Kalugin, D. J. Jeffrey, R. M. Corless, and P. B. Borwein. Stieltjes and other integral representations for functions of Lambert W. *Integral Transf. Spec. Funct.*, 23(8):581–593, 2012.

[104] W. Kapteyn. Recherches sur les fonctions de Fourier-Bessel. *Ann. Sci. de l'École Norm. Sup.*, 3(10):91–120, 1893.

[105] G. Keady. pers. comm. on nov. 9, 2015.

[106] C. Kittel. *Introduction to Solid State Physics (8th ed.)*. Wiley, 2004.

[107] R. A. Knoebel. Exponentials reiterated. *Amer. Math. Monthly*, 88:235–252, 1981.

[108] J. H. Lambert. Observationes variae in mathesin puram. *Acta Helveticae physico-mathematico-anatomico-botanico-medica, Band III (1758), 128-168. Available online at* http://www.kuttaka.org/~JHL/L1758c.html.

[109] J. H. Lambert. Observations analytiques. *Nouveaux Mémoires de l'Académie Royale des Sciences de Berlin, année 1770/1772, 225-244. Available online at* http://www.kuttaka.org/~JHL/L1770k.html.

[110] S. Lang. *Complex Analysis (4th ed.)*. Springer-Verlag, 1999.

[111] I. Langmuir. Positive ion currents from the positive column of mercury arcs. *Science*, 58:290–291, 1923.

[112] D. F. Lawden. *Elliptic Functions and Applications*. Springer-Verlag, 1989.

[113] B. Y. Levin. *Distribution of Zeros of Entire Functions ((revised ed.)*. Amer. Math. Soc., 1972.

[114] B. Y. Levin. *Lectures on Entire Functions*. Amer. Math. Soc., 1996.

[115] D. V. Lindley. Fiducial distributions and Bayes' theorem. *J. Royal Stat. Soc. Series B*, 20:102–107, 1958.

[116] D. V. Lindley. *Introduction to Probability and Statistics from a Bayesian Viewpoint, Part II: Inference*. Cambridge Univ. Press, 1965.

[117] Yu Liu. Collisions of the discrete Lambert map. *poster, Summer Undergraduate Math Research at Yale.*

[118] L. Lovász. *Combinatorial Problems and Exercises*. North Holland, 1993.

[119] W. A. J. Luxemburg. Banach Function Spaces. *PhD thesis, Technische Universiteit Delft, 1955.*

[120] M. K. Mahmood and L. Anwar. Loops in digraphs of Lambert mapping

modulo prime powers: enumerations and applications. *Adv. Pure Math.*, 6:564–570, 2016.

[121] A. Maignan, L. Prabhat Reddy, S. Jeevanandam, P. C. Deshmukh, K. Roberts, N. Jisrawi, and S. R. Valluri. The electronic properties of graphene nanoribbons and the offset logarithm function. *Materials Today: Proceedings (in press).*

[122] R. B. Mann and T. Ohta. Exact solution for the metric and the motion of two bodies in $(1+1)$-dimensional gravity. *Phys. Rev. D*, 55(8):4723–4747, 1997.

[123] W. S. Massey. *Algebraic Topology: An Introduction.* Springer, 1990.

[124] M. W. McCoy and B. M. Bolker. Trait-mediated interactions: influence of prey size, density and experience. *J. Animal Ecol.*, 77:478–586, 2008.

[125] S. G. Merzlyakov. Sharp estimates for the modulus of a canonical product. *Sbornik: Mathematics*, 207(2):238–266, 2016.

[126] I. Mező. On the discrete r-Lambert map. *unpublished manuscript.*

[127] I. Mező. The p-adic Lambert W function. *arXiv:1801.00657, 2018.*

[128] I. Mező. The r-Bell numbers. *J. Integer Seq.*, 14(1):Article 11.1.1, 2011.

[129] I. Mező. On the structure of the solution set of a generalized Euler-Lambert equation. *J. Math. Anal. Appl.*, 455:538–553, 2017.

[130] I. Mező. *Combinatorics and Number Theory of Counting Sequences.* CRC Press, 2020.

[131] I. Mező. The Riemann surface of the r-Lambert function. *Acta Math. Hung.*, 164(2):439–450, 2021.

[132] I. Mező and Á. Baricz. On the generalization of the Lambert W function. *Trans. Amer. Math. Soc.*, 369(11):7917–7934, 2017.

[133] I. Mező, C. B. Corcino, and R. B. Corcino. Resolution of the plane-symmetric Einstein – Maxwell fields with a generalization of the Lambert W function. *J. Phys. Commun.*, 4:Article no. 085008, 2020.

[134] L. Michaelis and M. L. Menten. Die kinetik der invertinwirkung. *Biochem. Z.*, 49:333–369, 1913.

[135] J. M. Mitchison and W. S. Vincent. A method of making synchronous cell cultures by density gradient centrifugation. In *Cell synchrony. Studies in biosynthetic regulation.* Academic Press, 1966.

[136] L. Moroni. The strange properties of the infinite power tower. *arXiv:1908.05559, 2019.*

[137] L. Moser and M. Wyman. An asymptotic formula for the Bell numbers. *Trans. Royal Soc. Canada*, 49:49–54, 1955.

[138] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim. The electronic properties of graphene. *Rev. Modern Phys.*, 81:109–162, 2009.

[139] A. H. Nuttall. A conjectured definite integral. Problem 85-16. *SIAM Review*, 27:573, 1985.

[140] O. Olendski. Theory of the Robin quantum wall in a linear potential. I. Energy spectrum, polarization and quantum-information measures. *Ann. Phys.*, 528(11-12):865–881, 2016.

[141] O. Olendski. Theory of the Robin quantum wall in a linear potential. II. Thermodynamic properties. *Ann. Phys.*, 528(11-12):882–897, 2016.

[142] O. Olendski. Thermodynamic properties of the 1D Robin quantum well. *Ann. Phys.*, 530(8):Article no. 1700325, 2018.

[143] E. W. Packel and D. S. Yuen. Projectile motion with resistance and the Lambert W function. *College Math. J.*, 35(5):337–350, 2004.

[144] G. Pólya and G. Szegő. *Aufgaben und Lehrsätze der Analysis I.* Springer, 1925. Reprinted as Problems and Theorems in Analysis I, Springer, 1998.

[145] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical Recipes: The Art of Scientific Computing (3rd ed.).* 2007.

[146] A. D. Rich and D. J. Jeffrey. Function evaluation on branch cuts. *SIGSAM Bulletin*, 116:25–27, 1996.

[147] K. U. Riemann. The Bohm criterion and sheath formation. *J. Phys. D.*, 24:493–518, 1991.

[148] D. Rogers. Random search and insect population models. *Journal of Animal Ecology*, 41(2):369–383, 1972.

[149] S. Roman. *Umbral Calculus.* Academic Press, 1984.

[150] B. Rosenbaum and B. C. Rall. Fitting functional responses: Direct parameter estimation by simulating differential equations. *Methods Ecol. Evol.*, 9:2076–2090, 2018.

[151] S. I. Rubinow. *Introduction to Mathematical Biology.* Wiley, 1975.

[152] B.-Z. A. Rubshtein, G. Ya. Grabarnik, M. A. Muratov, and Y. S. Pashkova. *Foundations of Symmetric Spaces of Measurable Functions.* Springer, 2016.

[153] H. Rutishauser. Anwendungen des Quotienten-Differenzen-Algorithmus. *Z. Angew. Math. Phys.*, 5:496–508, 1954.

[154] H. Rutishauser. Der Quotienten-Differenzen-Algorithmus. *Z. Angew. Math. Phys.*, 5:233–251, 1954.

[155] H. Rutishauser. Ein infinitesimales Analogon Zum Quotienten-Differenzen-Algorithmus. *Arch. Math.*, 5:132–137, 1954.

[156] S. Saha and K. Bamba. The Lambert W function: A newcomer in the cosmology class? *Z. Naturforsch.*, 75(1a):23–27, 2020.

[157] R. Schilling. *Bernstein Functions: Theory and Applications*. De Gruyter, 2010.

[158] S. Schnell and C. Mendoza. Closed form solution for time-dependent enzyme kinetics. *J. Theor. Biol.*, 187:207–212, 1997.

[159] T. C. Scott, M. Aubert-Frécon, and J. Grotendorst. New approach for the electronic energies of the hydrogen molecular ion. *Chem. Phys.*, 324:323–338, 2006.

[160] T. C. Scott, J. F. Babb, A. Dalgarno, and J. D. Morgan III. The calculation of exchange forces: General results and specific models. *J. Chem. Phys.*, 99:2841–2854, 1993.

[161] T. C. Scott, J. F. Babb, A. Dalgarno, and J. D. Morgan III. Resolution of a paradox in the calculation of exchange forces for H_2^+. *Chem. Phys. Lett.*, 203:175–183, 1993.

[162] T. C. Scott, R. Mann, and R. E. Martinez II. General relativity and quantum mechanics: towards a generalization of the Lambert W function. *Appl. Algebra Engrg. Comm. Comput.*, 17:41–47, 2006.

[163] I. H. Segel. *Enzyme Kinetics*. John Wiley & Sons, 1993.

[164] D. L. Shell. On the convergence of infinite exponentials. *Proc. Amer. Math. Soc.*, 13(5):678–681, 1962.

[165] H. Shinozaki. Lambert W Function Approach to Stability and Stabilization Problems for Linear Time-Delay Systems. *PhD Thesis, Kyoto Institute of Technology, 2007.*

[166] P. Song, H. J. Singer, and G. L. Siscoe (editors). Space Weather. In *Geophysical Monograph Series*. American Geophysical Union, 2001.

[167] M. Abramowitzand I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, 1972.

[168] S. M. Stewart. An analytic approach to projectile motion in a linear

resisting medium. *Internat. J. Math. Ed. Sci. Tech.*, 37(4):411–431, 2006.

[169] S. M. Stewart. On certain inequalities involving the Lambert W function. *J. Ineq. Pure Appl. Math.*, 10(4):Article 96., 2009.

[170] S. M. Stewart. On the trajectories of projectiles depicted in early ballistic woodcuts. *Eur. J. Phys.*, 33:149–166, 2012.

[171] H. U. Sverdrup. On conditions for the vernal blooming of phytoplankton. *Journal du Conseil International Pour l'Exploration de la Mer*, 18:287–295, 1953.

[172] P. L. Tchebycheff. Sur les fractions continues. *J. Math. Pures Appl. Ser. II*, 3:289–323, 1858.

[173] W. R. Thompson. La théorie mathématique de l'action des parasites entomophages et le facteur du hasard. *Annls. Fac. Sci. Marseille*, 2:69–89, 1924.

[174] W. J. Thron. Convergence of infinite exponentials with complex elements. *Proc. Amer. Math. Soc.*, 8(6):1040–1043, 1957.

[175] S. R. Valluri, M. Gil, D. J. Jeffrey, and S. Basu. The Lambert W function and quantum statistics. *J. Math. Phys.*, 50:Article 102103, 2009.

[176] S. R. Valluri, D. J. Jeffrey, and R. M. Corless. Some applications of the Lambert function to physics. *Can. J. Phys.*, 78(9):823–831, 2000.

[177] J. Vanderlinde. *Classical Electromagnetic Theory.* John Wiley & Sons, 1993.

[178] H. Vazquez-Leal, M. A. Sandoval-Hernandez, J. L. Garcia-Gervacio, A. L. Herrera-May, and U. A. Filobello-Nino. PSEM approximations for both branches of Lambert W function with applications. *Discrete Dyn. Nat. Soc.*, 2019:Article 8267951., 2019.

[179] E. B. Van Vleck. On the convergence of algebraic continued fractions whose coefficients have limiting values. *Trans. Amer. Math. Soc.*, 5:253–262, 1904.

[180] Ž. Kovač, T. Platt, and S. Sathyendranath. Sverdrup meets Lambert: analytical solution for Sverdrup's critical depth. *ICES Journal of Marine Science (2021), fsab013.*

[181] H. S. Wall. *Analytic Theory of Continued Fractions.* Chelsea Publishing Company, 1948.

[182] G. N. Watson. *A Treatise on the Theory of Bessel Functions (2nd ed.).* Cambridge Univ. Press, 1944.

[183] H. Weyl. *The Concept of a Riemann Surface (3rd ed.)*. Addison-Wesley, 1955.

[184] H. Wilf. The asymptotic behavior of the Stirling numbers of the first kind. *J. Combin. Th. Ser. A*, 64:344–349, 1993.

[185] E. M. Wright. The linear difference-differential equation with constant coefficients. *Proc. Royal Soc. Edinburgh Ser A*, 62:387–393, 1949.

[186] E. M. Wright. A non-linear difference-differential equation. *J. Reine Angew. Math.*, 194:66–87, 1955.

[187] E. M. Wright. Solution of the equation $ze^z = a$. *Proc. Roy. Soc. Edinburgh Ser. A*, 65:193–203, 1959.

[188] E. M. Wright. The number of connected sparsely edged graphs. *J. Graph Theory*, 1:317–330, 1977.

[189] S. Yi, P. W. Nelson, and A. G. Ulsoy. *Time-Delay Systems: Analysis and Control Using the Lambert W Function*. World Scientific, 2010.

[190] C.-Y. Zhu and A. Waldo. The discrete Lambert map. *RHIT Undergrad. Math. J.*, 16(2):181–194, 2015.

[191] D. Zirlin. Problems Motivated by Cryptology: Counting Fixed Points and Two-cycles of the Discrete Lambert Map. *BsC Thesis, Mount Holyoke College, 2015.*

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

Index

W as an inverse function, 4
$W'(x)$, 14
$W(x) + W(y)$, 12
$W(x) - W(y)$, 12
$W(x \cdot \log x)$, 11
W_b (base b Lambert function), 5, 8
Ω constant, 12, 30, 136
$\mathcal{K}$ function, 74, 190
$\mathfrak{M}_{m,n}$, 88
$\log(W(x))$, 10
ω function, 83
$aW(x) + bW(y)$, 11, 78
g-factor, 174
p_n polynomials, 15, 17, 38
qd-algorithm, 112
r-Lambert function
    definition, 127
$x + \log_b x = a$, 7, 82
$x + b^x = a$, 6, 173
$x \cdot \log_b x = a$, 7, 82
$x^x = a$, 9
$x^y = y^x$, 9
$xb^x = a$, 5
1D infinite potential well, 178

accelerating Parker solar wind
        solution, 184
algebraic number, 31, 134
algebraic topology, 70
angle trisection problem, 56
ansatz, 231
argument function, 41
argument principle, 98
associate space (of a Banach space),
        222
associated continued fraction, 113
asymptotics
    of W_r, 166
attenuation coefficient, 202

Baker – Bocharov – Paul – Rihan
        model, 197
Banach space, 220
barionic matter, 186
base b Lambert function, 5, 8
base space, 47
base-change formula, 5
Beer – Lambert law, 202
Bell number, 218
Bernstein function, 38
biomass growth, 202
black hole, 195
Bohm criterion, 191
Bohr magneton, 174
Boltzmann constant, 174, 179, 183,
        186
Bondi accretion solution, 184
boundary condition
    Dirichlet, 178
    Neumann, 178
    Robin, 178
branch closure
    of log, 42
branch cut, 44
branch difference, 88
branch index, 41
branch point, 28, 44
Brillouin function, 174

C-fraction
    convergence, 109
    finite, 108
    infinite, 108

non-terminating, 108
regular, 108
terminating, 108
Cayley's theorem, 216
CCC (counter-clockwise continuity), 44, 59, 65, 147
channel flow model, 181
characteristic equation (of a DDE), 231, 232
chromosphere, 184
coefficient of air resistance (k), 171
Colebrook – White equation, 195
collisional presheath, 192
commutative diagram, 52
conjugate function (Orlicz spaces), 222
continued fraction
element, 109
corona (of the sun), 184
correspondence
order of, 109
correspondence (between the continued fraction approximant and the represented function), 109
Cosmology, 195
counter-clockwise continuity (CCC), 44, 59, 147
counting function (of a sequence), 229
covering map, 47

De Moivre's formula, 40
Debye length, 191
defining equation, 3
delay differential equation, 197, 230
neutral, 197
derivative of W, 14
Dinostratus, 56
Dirichlet boundary condition, 178
discrete Lambert congruence, 131
discrete Lambert map, 131
doubling line, 84

Einstein model, 181
Einstein – Johnson sidewall model, 181
element
of a continued fraction, 109
entire function, 39
Euler formula, 31, 34
Eulerian number of the second kind, 19
exponential generating function, 216
exponential growth model, 198

fiber, 47
fiducial distribution, 205
finite C-fraction, 108
Formula for $aW(x) + bW(y)$, 11, 78
Free fall, 169
Friedmann – Lemaitre – Robertson – Walker model, 195
function space, 220

Gamma function, 23
generalized Laguerre polynomials, 128
generating function, 17, 216
Gragg's algorithm, 115
graph, 214
Great Picard Theorem, 40
Gumbel distribution, 207

Hölder's inequality, 221
Halley method, 105, 227
Hankel determinant, 111
Hankel determinant approach, 111
Hausdorff space, 47
Hippias of Elis, 56
hyperbolic cosecant, 90
hyperbolic cotangent, 90

incomplete Gamma function, 21, 25, 141
inequalities
for W, 121
infinite C-fraction, 108

Jacobi theta function, 179

Kapteyn series, 223
Kapteyn – Bessel series of the first type, 223

L'Hospital's rule, 33
Lady Gaga, 208
Lagrange inversion theorem, 25, 99, 128, 176
Laguerre polynomial, 128, 177
Langevin function, 174
Laplace limit constant, 176, 224
Laplace transform, 24
Lindemann – Weierstrass theorem, 31, 135
Lindley distribution, 206
Lorentz space, 220
lower parameter (of gen. Lambert function), 124

magnetization, 173
magnetization of a paramagnet, 173
Maple software, 95
Mathematica code
  qd-algorithm, 112
  Gragg's algorithm, 115
  how to get from the author, xiv
matrix Lambert function, 131, 232
maximum modulus principle, 218
maximum of standard normal variables, 207
Mellin transform, 23
Michaelis – Menten equation, 204
monodromy group, 53, 65
  of the logarithm, 53
monodromy groups of W, 65

Neumann boundary condition, 178
neutral delay differential equation, 197
Newtonian equation, 169
non-terminating C-fraction, 108
norming constant, 207

offset logarithm function, 194
omega
  single-valued, 83
order of correspondence, 109
Orlicz function, 221
Orlicz space, 221

Padé approximant, 105
paramagnet, 173
  magnetization, 173
Parker critical point, 183
partition function, 178
phytoplankton growth, 201
Picard's theorem, 39
Planck constant, 178, 186
polynomial-exponential type equation, 123
Prandtl's friction law, 181
preferential attachment rule, 209
presheath, 191
primitive root, 131
principal branch
  of W, 27
  of the logarithm, 41

quadratic Lambert function, 131, 195
quadratrix, 56
quaternion, 131
quaternionic Lambert equation, 131
Quotienten-Differenzen-Algorithmus, 112

Random Parasite Equation, 200
Random Predator Equation, 201
rational-exponential type equation, 123
real branches of W, 27
regular C-fraction, 108
Residue Theorem, 32
resistant media, 171
retweet graph, 208
Reynolds number, 181
Riemann surface, 48
  of the Lambert W function, 63
  of the logarithm, 50, 52
Riemann surface of the Lambert W function, 63
Riemann surface of the logarithm, 50, 52

Robin boundary condition, 178
Rodrigues formula, 128
Rouché's theorem, 98

S-fraction, 108
Schrödinger equation, 178, 193
shear stress, 181
sheath, 191
sheet, 47
sidewall correction methods, 181
sidewall friction factor, 181
special values, 12, 13
    of W, 4
    of W_r, 133
Spitzer – Härm heat conductivity, 182
squaring the circle, 56
stable solution of a DDE, 231
standard normal distribution, 207
Stirling number of the first kind, 100
Stirling polynomial, 100
String Theory, 195
superstar vertex, 209
surface, 47

terminating C-fraction, 108
tetration, 213
time delay, 231
time-lag growth model, 198
total space, 47
transcendental number, 31, 134
traveling time, 172
tree (graph theory), 215
trisectrix, 56
Twitter, 208
type (of the gen. Lambert function), 124

unpaired electron, 173
unwinding number, 74, 190
upper parameter (of gen. Lambert function), 124

Weierstrass primary factor, 229
Wien displacement law, 186
Wien's displacement constant, 186
Wright ω function, 83
Wright omega function, 83